

Liouville first-passage percolation: subsequential scaling limits at high temperature

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Abstract

Let $\{Y_{\mathfrak{B}}(v) : v \in \mathfrak{B}\}$ be a discrete Gaussian free field in a two-dimensional box \mathfrak{B} of side length S with Dirichlet boundary conditions. We study the Liouville first-passage percolation, in which each vertex is given a weight of $e^{\gamma Y_{\mathfrak{B}}(v)}$ for some $\gamma > 0$. We show that for sufficiently small but fixed $\gamma > 0$, for any sequence of scales $\{S_k\}$ there exists a subsequence along which the appropriately scaled Liouville FPP metric converges in the Gromov–Hausdorff sense to a random metric on the unit square in \mathbf{R}^2 . In addition, all possible (conjecturally unique) scaling limits are homeomorphic by bi-Hölder-continuous homeomorphisms to the unit square with the Euclidean metric.

1 Introduction

We consider Liouville first-passage percolation; i.e., first-passage percolation on the exponential of the discrete Gaussian free field. Given a box $\mathfrak{B} \subset \mathbf{Z}^2$ define $\overline{\mathfrak{B}}$, the *blow-up* of \mathfrak{B} , as the box of three times the side length centered around \mathfrak{B} . What we will call the discrete Gaussian free field on \mathfrak{B} is the restriction to \mathfrak{B} of the standard discrete Gaussian free field with Dirichlet boundary conditions on $\overline{\mathfrak{B}}$. This is the mean-zero Gaussian process $Y_{\mathfrak{B}}(x)$ such that $Y_{\mathfrak{B}}(x) = 0$ for all $x \in \partial \overline{\mathfrak{B}}$ and $\mathbf{E}Y_{\mathfrak{B}}(x)Y_{\mathfrak{B}}(y) = G_{\overline{\mathfrak{B}}}(x, y)$ for all $x, y \in \mathfrak{B}$, where $G_{\overline{\mathfrak{B}}}(x, y)$ is the Green’s function of simple random walk on $\overline{\mathfrak{B}}$. (Note that the constant 3 in the definition of the blow-up is irrelevant to the result—the point is that Dirichlet boundary conditions are imposed on a box which is a constant fraction larger.)

Fix an inverse-temperature parameter $\gamma > 0$. Let Y_S denote the discrete Gaussian free field on $\mathfrak{B}_S = [0, S]^2 \cap \mathbf{Z}^2$. We define the *Liouville first-passage percolation* metric d_S on \mathfrak{B}_S by

$$d_S(x_1, x_2) = \min_{\pi} \sum_{x \in \pi} e^{\gamma Y_S(x)},$$

where π ranges over all paths in \mathfrak{B}_S connecting x_1 and x_2 . Given a sequence of normalizing constants κ_S , we define a metric \tilde{d}_S on $[0, 1]^2 \subset \mathbf{R}^2$ by letting

$$\tilde{d}_S(x_1, x_2) = \frac{1}{\kappa_S} d_S(Sx_1, Sx_2)$$

for each $x_1, x_2 \in [0, 1]^2 \cap \frac{1}{S}\mathbf{Z}^2$ and extending to all $x_1, x_2 \in [0, 1]^2$ by linear interpolation. We will prove the following.

Theorem 1.1. *There is a $\gamma_0 > 0$ so that if $\gamma < \gamma_0$ then there exists a sequence of normalizing constants κ_S so that, for every sequence of scales S_i , there is a subsequence $\{S_{i_j}\}$ so that $\tilde{d}_{S_{i_j}}$ converges in distribution (using the Gromov–Hausdorff topology on the space of metrics) to a limiting metric, which moreover is homeomorphic to the Euclidean metric by a Hölder-continuous homeomorphism with Hölder-continuous inverse.*

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1.1 Background and related results

Much effort (see [ADH15, GK12] and their references) has been devoted to understanding classical first-passage percolation, with independent and identically distributed edge/vertex weights. We argue that FPP with strongly-correlated weights is also a rich and interesting subject, involving questions both analogous to and divergent from those asked in the classical case. Since the Gaussian free field is in some sense the canonical strongly-correlated random medium, we see strong motivation to study Liouville FPP (FPP in the plane where the weights are given by the exponential of DGFF).

More specifically, Liouville FPP is thought to play a key role in understanding the random metric associated with the Liouville quantum gravity (LQG) [Pol81, DS11, RV14]. Even just to make rigorous sense of the random metric of LQG is a major open problem. Recently, Miller and Sheffield have made substantial progress in the continuum setup for the case $\gamma = \sqrt{8/3}$; see [MS15b, DMS14, MS15a] and their references. In these papers, the authors focused more on directly constructing the random metric in the continuum setup. We take an alternative approach which aims at understanding the random metric of LQG via scaling limits of natural discrete approximations. We choose to work with Liouville FPP both for its simple formulation and for its connection to classical FPP. We note that eventually one might wish to tweak the definition of the discrete metric in order to obtain a conformally invariant scaling limit. However, we believe the methods developed in this article provide a robust framework for studying random metrics similar to Liouville FPP.

Our result is similar in flavor to [LG07] and [LGP08], which proved, respectively, that the graph distance of random quadrangulations has a subsequential scaling limit and that the all possible limiting metrics are homeomorphic to a 2-sphere. (In our case, however, the homeomorphism property is a byproduct of the compactness result.) The uniqueness of the scaling limit, known as the Brownian map, was proved in later works [LG10, LG13, Mie13]. A crucial ingredient in [LG07] is a bijection [CV81, Sch88, BDFG04] between uniform quadrangulations and labeled trees. In particular, such a bijection allows an explicit evaluation of the order of the typical distance in the random quadrangulation.

In our model, determining the FPP weight exponent seems to be a major challenge. Indeed, in recent works [DG15, DG] it was shown that the weight exponent is strictly less than 1 at high temperatures, and in [DZ15] it was shown that there exists a family of log-correlated Gaussian fields where the weight exponent can be arbitrarily close to 1. This means that the weight exponent is not universal among log-correlated Gaussian fields, which in turn indicates substantial subtlety and difficulty inherent in a precise computation of this exponent. Our proof circumvents this issue by avoiding explicit computation of the order of the typical distance.

1.2 Proof approach and the RSW method

The framework of our proof (which we note bears little similarity to the methods used in [DG15, DG]) is a multiscale analysis procedure relying on several relationships which we establish between FPP weights at different scales. The key estimates are inductive upper and lower bounds on crossing weights and geodesic lengths, in which weights and lengths at a larger scale are estimated in terms of weights at a smaller scale. Most of the lower bounds on the larger-scale weights are achieved in Section 4 using percolation-type arguments, while the upper bounds on larger-scale weights and lengths are carried out in Section 6 using gluing arguments along with the lower bounds. In Subsection 6.3, we use a chaining argument to get an upper bound on box *diameter*, which combined with the lower bounds allows us to inductively bound the crossing weight coefficient of variation in Section 7. Finally, in Section 8, we apply this coefficient of variation bound to establish tightness, and thus subsequential convergence, of the normalized FPP metrics.

Carrying out the above strategy leads to a central problem: lower bounds on crossing weights are obtained in terms of “easy crossings” (between the two longer sides) of rectangles, while upper bounds are obtained in terms of “hard crossings” (between the two shorter sides). (See Figure 1.1) In order to play these bounds off of each other, we must establish a relationship between easy and hard crossing weights. Results of this type are known as RSW

statements, and a substantial part of our paper (Section 5) is dedicated to establishing such a result in the Liouville FPP setting.

We briefly review the history of the RSW method, an important technique in planar statistical physics, initiated in [Rus78, SW78, Rus81] in order to prove a positive crossing probability through a rectangle in critical Bernoulli percolation. Recently, an RSW theory has been developed for FK percolation; see e.g. [DCHN11, BDC12, DCST15]. In [Tas14], an RSW theory was developed for Voronoi percolation. In fact, the beautiful method in [Tas14] is widely applicable to percolation problems satisfying the FKG inequality, mild symmetry assumptions, and weak correlation between well-separated regions. For example, in [DCRT16], this method was used to give a simpler proof of the result of [BDC12], and in [DCMT], the authors proved an RSW theorem for the crossing probability of level sets of planar Gaussian free field.

Our RSW proof is *hugely* inspired by [Tas14]. A main novelty of our result is that it seems to be the first RSW theorem for random planar metrics (other than traditional crossing probabilities for various percolation problems). We believe that the applicability of RSW theory in the metric setting may have a chance to enrich both the application and the theory of RSW method, and we expect more applications of RSW theory in the study of random planar metrics. The introduction of the FPP weight in our RSW result incur substantial challenges even given the beautiful work of [Tas14]: the proof method of [Tas14] is based on an intrinsic induction which becomes far more delicate with the FPP weight taken into account. Besides that, our FPP metric is lacking of a natural self-duality, which precludes using the hypothesis of crossing square boxes as in the traditional setup; rather, we start with “easy” crossings of rectangular boxes. The difficulties are such that we were only able to relate *different* quantiles of the FPP weight in different scales, and we have to apply our induction hypothesis on the variance of the FPP weight to relate different quantiles at a given scale. This introduces an additional layer of complexity to our arguments.

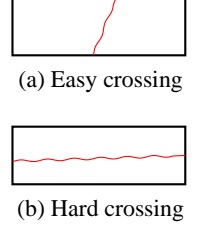


Figure 1.1

1.3 Acknowledgment

We thank Steve Lalley for encouragement and useful discussions.

2 Preliminaries

2.1 Notational conventions

Since we will be primarily working in the discrete setting, throughout the paper, the notation $[a, b)$ will denote the set of integers between a and $b - 1$, inclusive, and $[a, b]$ the set of integers between a and b , inclusive. When we need to refer to an interval of real numbers, we will attach a subscript \mathbf{R} , as in $[a, b]_{\mathbf{R}}$, etc. A *box* is a rectangular subset of \mathbf{Z}^2 . As in the introduction, the *blow-up* of a box \mathfrak{B} is the union of the nine translates of \mathfrak{B} centered around \mathfrak{B} . We say that a rectangular box is *portrait* if its height is greater than its width and *landscape* if its width is greater than its height. For boxes $\mathfrak{U} \subseteq \mathfrak{B}$, we will use the notation $|\mathfrak{B}/\mathfrak{U}|$ to denote the maximum of the width of \mathfrak{B} divided by the width of \mathfrak{U} and the height of \mathfrak{B} divided by the height of \mathfrak{U} .

Suppose π is a path and Y is a random field. Define

$$\psi(\pi; Y) = \sum_{x \in \pi} \exp(\gamma Y(x)).$$

If \mathfrak{R} is a rectangle, let

$$\Psi_{\text{LR}}(\mathfrak{R}; Y) = \min_{\pi} \psi(\pi; Y),$$

where π ranges over all left–right crossings of \mathfrak{R} . Define Ψ_{BT} analogously for bottom–top crossings. Also put

$$\Psi_{\text{easy}}(\mathfrak{R}; Y) = \min_{\pi} \psi(\pi; Y)$$

where π ranges over all crossings between the longer sides of \mathfrak{R} , and let

$$\Psi_{\text{hard}}(\mathfrak{R}; Y) = \min_{\pi} \psi(\pi; Y)$$

where π ranges over all crossings between the shorter sides of \mathfrak{R} . (Hence $\pi_{\text{easy}}(\mathfrak{R}; Y) = \pi_{\text{LR}}(\mathfrak{R}; Y)$ if \mathfrak{R} is portrait, etc.) Moreover, let

$$\begin{aligned} \Psi_{x,y}(\mathfrak{B}; Y) &= \min_{\pi: x \rightsquigarrow y} \psi(\pi; Y) \text{ for } x, y \in \mathfrak{B}, \\ \Psi_{\partial}(\mathfrak{B}; Y) &= \max_{x, y \in \partial \mathfrak{B}} \Psi_{x,y}(\mathfrak{B}; Y), \text{ and} \\ \Psi_{\max}(\mathfrak{B}; Y) &= \max_{x, y \in \mathfrak{B}} \Psi_{x,y}(\mathfrak{B}; Y). \end{aligned}$$

In all of these notations, we have defined $\Psi_{\bullet}(\mathfrak{B}; Y)$ as the minimum of $\psi(\cdot; Y)$ over a collection of paths. In each case let $\pi_{\bullet}(\mathfrak{B}; Y)$ be the path that achieves the minimum; if there are multiple paths, choose one uniformly at random. We also need notation for the quantile functions for these variables, so let

$$\Theta_{\bullet}(\mathfrak{B}; Y)[p] = \inf\{w \mid \mathbf{P}[\Psi_{\bullet}(\mathfrak{B}; Y) \leq w] \geq p\}.$$

For a path π , let $|\pi|$ denote the length of π . Let $\|\pi\|_S$ denote the number of dyadic square boxes of side-length S entered by π , counting each box *once*, even if π enters it multiple times. Let $M_{\bullet, S}(\mathfrak{R}; Y) = \|\pi_{\bullet}(\mathfrak{R}; Y)\|_S$.

Whenever the field is omitted in the Ψ or Θ notation, it will be assumed to be the Gaussian free field on the box in question, defined as in the introduction as the discrete Gaussian free field with Dirichlet boundary conditions on the boundary of the box.

Big- O and little- o notation will be employed, always with the limit taken as $\gamma \rightarrow 0$. Subscripts will be employed to indicate that the limit holds for any *fixed* value of the variable(s) in the subscript, and uniformly in all other variables. Most importantly, the limit is *always uniform* in the current scale. We will also work with many constants throughout the proofs. The important point regarding any constant is that it is independent of the scale. Constants that will be referred to between sections will be denoted by a mnemonic subscript.

2.2 The coarse field

Our multiscale analysis procedure will rely on comparing FPP in different boxes. We will adopt a (convenient but somewhat notationally abusive) convention that distinct Gaussian free fields will be considered as coupled (in a manner that we detail below) whenever this makes sense; see Remark 2.3 below.

In this section we provide precise statements and references for the fact that, when γ is small, the “course” part of the GFF (corresponding to long-range correlations) is negligible. This fact will be a cornerstone of our analysis and represents essentially the entirety of the technical dependence of our results on γ being small.

Lemma 2.1. *There is an absolute constant C_F so that the following holds. Let $Y_{\mathfrak{B}}$ be a Gaussian free field on \mathfrak{B} and let $\mathfrak{A} \subset \mathfrak{B}$. Then we have*

$$\mathbf{E} \left[\max_{x \in \mathfrak{A}} \mathbf{E} \left[Y_{\mathfrak{B}}(x) \mid Y_{\mathfrak{B}} \upharpoonright \partial \overline{\mathfrak{A}} \right] \right] < C_F.$$

Proof. This follows from Fernique’s criterion (see [Fer75] and [Adl90, Theorem 4.1]) and a covariance estimate on the conditional expectation field; see [BDZ16, Lemmas 3.5 and 3.10] for proof. \square

Lemma 2.2. *There is a constant C so that the following holds. Let $Y_{\mathfrak{B}}$ be a Gaussian free field on \mathfrak{B} and let $\mathfrak{A}_1, \dots, \mathfrak{A}_M \subset \mathfrak{B}$ be identically-sized subboxes. Then, for each i ,*

$$\mathbf{P} \left(\max_{x \in \mathfrak{A}_i} \left[\mathbf{E} Y_{\mathfrak{B}}(x) \mid Y_{\mathfrak{B}} \upharpoonright \partial \overline{\mathfrak{A}_i} \right] \geq C_F + u \right) \leq \exp \left(- \frac{Cu^2}{\log |\mathfrak{B}/\mathfrak{A}_i|} \right).$$

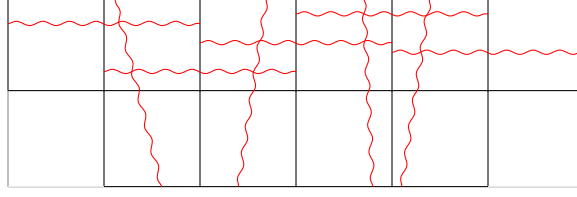


Figure 2.1: Gluing strategy in Corollary 2.5 for $a = 2b$ and $k = 5$.

Proof. This follows from Lemma 2.1, Theorem 7.1 of [Led01], and the fact that the variance of $\mathbf{E}[Y_{\mathfrak{B}}(x) \mid Y_{\mathfrak{B}} \upharpoonright \partial\overline{\mathfrak{A}}]$ can be bounded (uniformly over $x \in \mathfrak{A}$) by a constant times $\log|\mathfrak{B}/\mathfrak{A}|$. \square

For each $x \in \mathfrak{A}$, put

$$Y_{\mathfrak{A}}(x) = Y_{\mathfrak{B}}(x) - \mathbf{E}\left[\max_{x \in \mathfrak{A}} \left[\mathbf{E}Y_{\mathfrak{B}}(x) \mid Y_{\mathfrak{B}} \upharpoonright \partial\overline{\mathfrak{A}} \right] \right]. \quad (2.1)$$

Then, by Lemma 2.1, we have

$$\max_{x \in \mathfrak{A}} \left| \frac{e^{\gamma Y_{\mathfrak{B}}(x)}}{e^{\gamma Y_{\mathfrak{A}}(x)}} \right| = 1 + o(1) \quad (2.2)$$

in probability (as $\gamma \rightarrow 0$). In particular, by Lemma 2.2, there is an absolute constant $u_0 > 1$ so that if $u \geq u_0$ then we have

$$\mathbf{P}\left(\max_{x \in \mathfrak{A}} \left| \frac{e^{\gamma Y_{\mathfrak{B}}(x)}}{e^{\gamma Y_{\mathfrak{A}}(x)}} \right| \geq u\right) + \mathbf{P}\left(\max_{x \in \mathfrak{A}} \left| \frac{e^{\gamma Y_{\mathfrak{B}}(x)}}{e^{\gamma Y_{\mathfrak{A}}(x)}} \right| \leq \frac{1}{u}\right) = \mathbf{P}\left(\max_{x \in \mathfrak{A}} |\gamma Y_{\mathfrak{B}}(x) - \gamma Y_{\mathfrak{A}}(x)| \geq \log u\right) \leq \exp\left(-\Omega(1) \cdot \frac{(\log u)^2}{\log|\mathfrak{B}/\mathfrak{A}|}\right). \quad (2.3)$$

Moreover, if \mathfrak{A} and \mathfrak{A}' are two such boxes, then $\{Y_{\mathfrak{A}}(x)\}_{x \in \mathfrak{A}} \cup \{Y_{\mathfrak{A}'}(x)\}_{x \in \mathfrak{A}'}$ is a positively-correlated Gaussian process, and if $\overline{\mathfrak{A}} \cap \overline{\mathfrak{A}'} = \emptyset$, then $\{Y_{\mathfrak{A}}(x)\}_{x \in \mathfrak{A}}$ and $\{Y_{\mathfrak{A}'}(x)\}_{x \in \mathfrak{A}'}$ are independent.

Remark 2.3. Unless more specifically stated, when we discuss Gaussian free field on multiple rectangles at once, we will implicitly be referring to the field obtained by (2.1) from a single GFF on the smallest rectangle containing all of the rectangles.

We will frequently use the following version of the celebrated Fortuin–Kasteleyn–Ginibre (FKG) inequality, which has seen wide use in the study of percolation.

Theorem 2.4. *If f and g are increasing functions of a positively-correlated Gaussian field Y , then*

$$\mathbf{E}f(Y)g(Y) \geq \mathbf{E}f(Y)\mathbf{E}g(Y).$$

See [Pit82] for a proof.

Corollary 2.5. *Let $a > b$ and $S, k \in \mathbb{N}$, and put $\mathfrak{A} = [0, aS) \times [0, bS)$ and $\mathfrak{B} = [0, (ka - (k-1)b)S) \times [0, bS)$. Then*

$$\mathbf{P}[\Psi_{\text{LR}}(\mathfrak{B}) \leq 2ky] \geq \mathbf{P}[\Psi_{\text{LR}}(\mathfrak{A}) \leq y]^{2k-1} - o_k(1).$$

Proof. Follows from (2.2) and the FKG inequality as shown in Figure 2.1. \square

Corollary 2.6. *Let S and $b < k$ be natural numbers and put $\mathfrak{A} = [0, bS) \times [0, (b+1)S)$ and let $\mathfrak{B} = [0, bS) \times [0, kS)$. Then*

$$\mathbf{P}[\Psi_{\text{easy}}(\mathfrak{B}) \leq y] \leq 2k\mathbf{P}[\Psi_{\text{easy}}(\mathfrak{A}) \leq y] + o_k(1).$$

Proof. Divide the rectangle

$$[0, bS) \times [0, kS)$$

into k $bS \times S$ landscape subrectangles. Now a left–right crossing of $[0, bS) \times [0, kS)$ must either cross horizontally within a block of $b + 1$ of the rectangles ($k - b$ such blocks) such or cross a block of b of the rectangles vertically ($k - b + 1$ such blocks). Each of these events has probability at most

$$\mathbf{P}[\Psi_{\text{LR}}(\mathfrak{R}) \leq y] + o_{|\mathfrak{B}/\mathfrak{U}|}(1)$$

(using (2.2)) so the conclusion of the lemma follows from a union bound. \square

3 Inductive hypothesis

The key ingredient for all of our results is an inductive bound on the coefficient of variation for the FPP crossing weight of a rectangle.

Theorem 3.1. *Let $\delta > 0$. If γ is sufficiently small compared to δ , then for all boxes \mathfrak{R} of aspect ratio between $1/2$ and 2 inclusive, we have $\text{CV}^2(\Psi_{\text{LR}}(\mathfrak{R})) < \delta$.*

The bulk of the paper will be devoted to proving Theorem 3.1 using induction on the scale. Actually, we will have to use the slightly stronger inductive hypothesis that the coefficient of variation is below a fixed δ_0 . The following lemma, which is an easy consequence of Chebyshev’s inequality, will be key to our induction.

Lemma 3.2. *Let X and Y be nonnegative random variables. Put $F_X(x) = \mathbf{P}(X \leq x)$, $F_Y(y) = \mathbf{P}(Y \leq y)$. If $\text{CV}^2(X) < \delta < p$ and $\text{CV}^2(Y) < \varepsilon < q$, then there are constants $0 < A \leq B$, depending only on $\delta, \varepsilon, p, q$ (and not on the random variables X, Y) so that*

$$A \cdot \frac{F_X^{-1}(p)}{F_Y^{-1}(q)} \leq \frac{\mu_X}{\mu_Y} \leq B \cdot \frac{F_X^{-1}(p)}{F_Y^{-1}(q)}. \quad (3.1)$$

Suppose moreover that $\delta < p'$ and $\varepsilon < q'$. Then there are constants $0 < A' \leq B'$, depending only on $\delta, \varepsilon, p, q, p', q'$, so that

$$A' \cdot \frac{F_X^{-1}(p)}{F_Y^{-1}(q)} \leq \frac{F_X^{-1}(p')}{F_Y^{-1}(q')} \leq B' \cdot \frac{F_X^{-1}(p)}{F_Y^{-1}(q)}. \quad (3.2)$$

4 Crossing quantile lower bounds

Our goal in this section is to obtain lower bounds on quantiles of the left–right crossing weight of a large box in terms of the easy crossing quantiles of smaller boxes. We first define and introduce basic properties of what we call *passes*, which represent smaller boxes through which a path through a larger box must cross.

Let $K, L \geq 2$ and let $S = 2^s$. Let $\mathfrak{R} = [0, KS) \times [0, LS)$.

Definition 4.1. A *pass* of \mathfrak{R} at scale S is an $S \times S$, $2S \times S$, or $S \times 2S$ dyadic subrectangle of \mathfrak{R} .

Definition 4.2. Let π be a path in \mathfrak{R} . Given a pass \mathfrak{P} , we say that π *crosses* \mathfrak{P} if π connects the two longer sides of \mathfrak{P} (or any two opposite sides of \mathfrak{P} if \mathfrak{P} is a square) while staying within \mathfrak{P} .

Lemma 4.3. *Let π be a left–right crossing of \mathfrak{R} . If π enters an $S \times S$ box \mathfrak{C} such that $\overline{\mathfrak{C}} \subseteq \mathfrak{R}$, then π must cross a pass contained in $\overline{\mathfrak{C}}$.*

Proof. Since π is a left–right crossing of \mathfrak{R} , π must at some point leave $\overline{\mathfrak{C}}$. And it is easy to see that in order to cross the annulus $\overline{\mathfrak{C}} \setminus \mathfrak{C}$, π must cross a pass contained in $\overline{\mathfrak{C}}$. \square

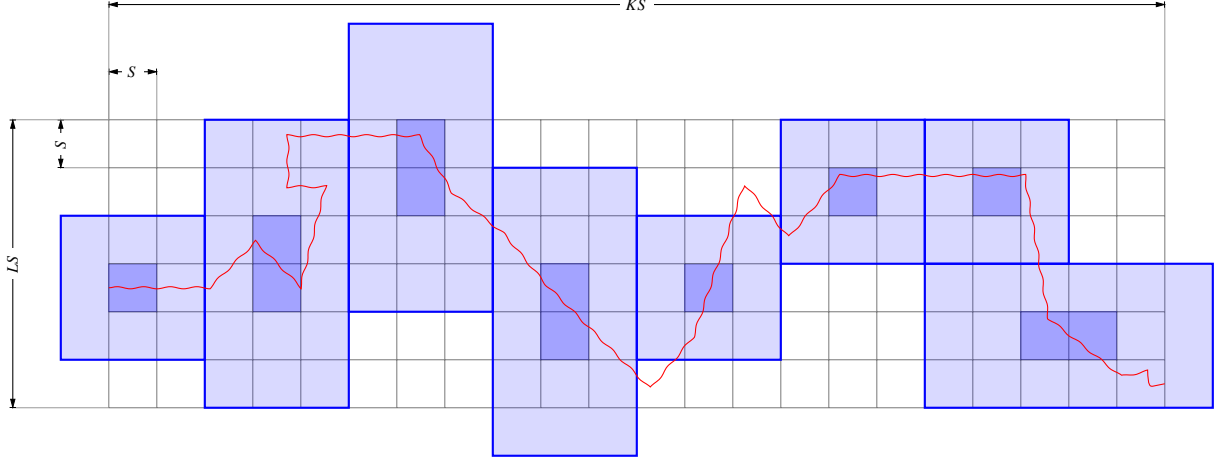


Figure 4.1: $\mathcal{P}(\pi)$ for a crossing π . The darker boxes are the \mathfrak{P}_i s while the lighter, surrounding boxes are the $\overline{\mathfrak{P}}_{i,S}$.

Definition 4.4. For a path π , let $\mathcal{P}(\pi)$ be a maximum-size collection of passes \mathfrak{P} crossed by π such that the $\overline{\mathfrak{P}}$ s are disjoint. (See Figure 4.1.) For $N \leq |\mathcal{P}(\pi)|$, let $\mathcal{P}_N(\pi) = \mathcal{P}(\xi)$ where ξ is the minimal initial subpath ξ of π such that $|\mathcal{P}(\xi)| \geq N$.

Proposition 4.5. There is a constant c_{PD} so that $|\mathcal{P}(\pi)| \geq c_{PD} \|\pi\|_S$. (The subscript stands for “pass density.”)

Proposition 4.6. For any path π and any $N \leq |\mathcal{P}(\pi)|$, we have

$$\psi(\pi; Y_{\mathfrak{R}}) \geq \sum_{\mathfrak{P} \in \mathcal{P}_N(\pi)} \Psi_{\text{easy}}(\mathfrak{P}; Y_{\mathfrak{R}}).$$

Lemma 4.7. If π is a left–right crossing of \mathfrak{R} , then $|\mathcal{P}(\pi)| \geq K/3$.

Proof. In order for π to cross each column of width S , it must cross a pass contained entirely within that column. \square

Lemma 4.8. Let G be a graph with maximum degree d , and let $\{a_1, \dots, a_M\}$ be an arbitrary subset of the vertices of G . Then the number of n -vertex connected subgraphs H of G containing at least one a_i is at most $M \cdot d^{2n}$.

Proof. It is easy to see that every graph on n vertices contains a circuit of length at most $2n$ that visits every vertex. Thus the number of subgraphs H as specified in the statement is bounded by the number of walks of length at most $2n$ starting at one of the a_i s, which is evidently bounded by $M \cdot d^{2n}$. \square

Proposition 4.9. We have a constant d_{pass} so that

$$|\{\mathcal{P}_N(\pi) : \pi \text{ a left–right crossing of } \mathfrak{R} \text{ such that } |\mathcal{P}(\pi)| \geq N\}| \leq 2L \cdot d_{\text{pass}}^{2N}.$$

Proof. Define a graph G on the set of all passes inside \mathfrak{R} by saying that two passes are adjacent if they could occur as adjacent passes in a $\mathcal{P}(\pi)$. It is easy to see using Lemma 4.3 that G has bounded degree. Then by definition, $\mathcal{P}_N(\pi)$ induces an N -element connected subgraph of G , which in particular contains a pass in the first column, of which there are $2L - 1$. Lemma 4.8 then implies the desired result. \square

Before we prove the main proposition of this section, we need a version of the Chernoff bound.

Lemma 4.10. Let $p < \frac{1}{2}$ and X_1, \dots, X_N be iid Bernoulli(p) random variables. Then $\mathbf{P}\left[\frac{1}{N} \sum_{i=1}^N X_i \geq \frac{1}{2}\right] \leq (4p)^{N/2}$.

Proof. We have $\mathbf{P}\left[\frac{1}{N} \sum_{i=1}^N X_i \geq \frac{1}{2}\right] = \mathbf{P}\left[\exp\left(\lambda \cdot \sum_{i=1}^N X_i\right) \geq e^{\lambda N/2}\right] \leq \frac{(\mathbf{E}e^{\lambda X_i})^N}{e^{\lambda N/2}} = \left(\frac{pe^{\lambda+1}-p}{e^{\lambda/2}}\right)^N$. Putting $\lambda = \log \frac{1-p}{p}$ and using the fact that $p < 1/2$ yields the result. \square

Now we can prove an inductive lower bound on the crossing weight.

Proposition 4.11. *Let $S = 2^s$ and let $K, L \in \mathbf{N}$. Let $\mathfrak{R} = [0, KS) \times [0, LS)$ and $\mathfrak{A} = [0, S) \times [0, 2S)$. Then, for any $p \in (0, 1/2)$ and any $u \geq u_0$, we have*

$$\mathbf{P}\left[\min_{\pi} \psi(\pi; Y_{\mathfrak{R}}) \leq \frac{N}{2u} \Theta_{\text{easy}}(\mathfrak{A})[p]\right] \leq KL \left[3(2d_{\text{pass}}^2 \sqrt{p})^N + \exp\left(-\Omega(1) \cdot \frac{(\log u)^2}{\log(K \vee L)}\right)\right],$$

where the minimum is taken over all paths π with $|\mathcal{P}(\pi)| \geq N$, and have

$$\mathbf{P}\left[\min_{\pi} \psi(\pi; Y_{\mathfrak{R}}) \leq \frac{N}{2u_0} \Theta_{\text{easy}}(\mathfrak{A})[p]\right] \leq 2L(2d_{\text{pass}}^2 \sqrt{p})^N + o_{K,L}(1),$$

where the minimum is taken over all left-right crossings π of \mathfrak{R} .

Proof. As long as γ is sufficiently small compared to K and L and $u \geq u_0$, we have, using (2.3) and a union bound,

$$\begin{aligned} \mathbf{P}\left[\psi(\pi; Y_{\mathfrak{R}}) \leq \frac{N}{2u} \Theta_{\text{easy}}(\mathfrak{A})[p]\right] &\leq \mathbf{P}\left[\frac{1}{N} \sum_{\mathfrak{P} \in \mathcal{P}_N(\pi)} \mathbf{1}\{\Psi_{\text{easy}}(\mathfrak{P}; Y_{\mathfrak{R}}) \leq \Theta_{\text{easy}}(\mathfrak{A})[p]\} \geq \frac{1}{2}\right] \\ &\leq \mathbf{P}\left[\frac{1}{N} \sum_{\mathfrak{P} \in \mathcal{P}_N(\pi)} \mathbf{1}\{\Psi_{\text{easy}}(\mathfrak{P}) \leq \Theta_{\text{easy}}(\mathfrak{A})[p]\} \geq \frac{1}{2}\right] + KL \exp\left(-\Omega(1) \cdot \frac{(\log u)^2}{\log(K \vee L)}\right) \\ &\leq \mathbf{P}\left[\frac{1}{N} \sum_{\mathfrak{P} \in \mathcal{P}_N(\pi)} \mathbf{1}\{\Psi_{\text{easy}}(\mathfrak{A}) \leq \Theta_{\text{easy}}(\mathfrak{A})[p]\} \geq \frac{1}{2}\right] + KL \exp\left(-\Omega(1) \cdot \frac{(\log u)^2}{\log(K \vee L)}\right) \\ &\leq (4p)^{N/2} + KL \exp\left(-\Omega(1) \cdot \frac{(\log u)^2}{\log(K \vee L)}\right). \end{aligned}$$

Then Proposition 4.9 and union bounds imply the results. \square

Corollary 4.12. *Fix a scale $S = 2^s$ and let $K, L \in \mathbf{N}$. Let $\mathfrak{R} = [0, KS) \times [0, LS)$ and $\mathfrak{A} = [0, S) \times [0, 2S)$. Then we have*

$$\Theta_{\text{LR}}(\mathfrak{R}) \left[2L(2d_{\text{pass}}^2 \sqrt{p})^{K/3} + o_{K,L,p}(1)\right] \geq \frac{K}{6u_0} \Theta_{\text{easy}}(\mathfrak{A})[p]$$

and

$$\mathbf{E}\Psi_{\text{LR}}(\mathfrak{R}) \geq \frac{K}{6u_0} \Theta_{\text{easy}}(\mathfrak{A})[p] \cdot \left(1 - 2L(2d_{\text{pass}}^2 \sqrt{p})^{K/2u_0} - o_{K,L,p}(1)\right).$$

Proof. If π is a path from left to right in \mathfrak{R} , then by Lemma 4.7, we have $|\mathcal{P}(\pi)| \geq K/3$. Proposition 4.11 then implies the first equation. The second equation follows immediately from the first. \square

We conclude this section with an inductive version of Corollary 4.12, showing that some easy crossing quantile grows like $S^{1-o(1)}$ in the scale S .

Proposition 4.13. *There are constants $p_{\text{pl}}, q_{\text{pl}}, a_{\text{pl}} \in (0, 1)$ and $K_{\text{pl}}, C_{\text{pl}} > 0$ so that, if $p < p_{\text{pl}}$ and $K \geq K_{\text{pl}}$, and γ is sufficiently small compared to p and K , then, putting $\mathfrak{R}_t = [0, 2^t) \times [0, 2^{t+1})$, for any $s > t$ we have*

$$\Theta_{\text{easy}}(\mathfrak{R}_t)[p_{\text{pl}}] \leq C_{\text{pl}} a_{\text{pl}}^{s-t} \Theta_{\text{easy}}(\mathfrak{R}_s)[q_{\text{pl}}].$$

(The subscript pl stands for “power-law.”)

Proof. Write $s = t + nk + r$, where $0 \leq r < k = \log_2 K$. Let $R = 2^r$. We can calculate, using Corollary 4.12,

$$\Theta_{\text{easy}}(\mathfrak{R}_t)[p] \leq \frac{6u_0}{K} \Theta_{\text{easy}}(\mathfrak{R}_{t+k}) \left[4K \left(d_{\text{pass}}^2 \sqrt{p} \right)^{K/3} + o_K(1) \right] \leq \frac{6u_0}{K} \Theta_{\text{easy}}(\mathfrak{R}_{t+k})[p],$$

where in the second inequality we use the assumption that p is sufficiently small, K is sufficiently large, and γ is sufficiently small (compared to p and K). By induction we obtain

$$\Theta_{\text{easy}}(\mathfrak{R}_t)[p] \leq \left(\frac{6u_0}{K} \right)^n \Theta_{\text{easy}}(\mathfrak{R}_{t+nk})[p] = \left(\frac{(6u_0)^{1/k}}{2} \right)^{kn} \Theta_{\text{easy}}(\mathfrak{R}_{t+nk})[p]$$

Thus, applying Corollary 4.12 once more, we get

$$\Theta_{\text{easy}}(\mathfrak{R}_{t+nk})[p] \leq \frac{6u_0}{R} \Theta_{\text{easy}}(\mathfrak{R}_{t+nk+r}) \left[4R \left(d_{\text{pass}}^2 \sqrt{p} \right)^{R/3} + o_R(1) \right] \leq \frac{6u_0}{R} \Theta_{\text{easy}}(\mathfrak{R}_{t+nk+r})[q_{\text{pl}}],$$

where p , K , γ are restricted so that q_{pl} can be chosen to be less than 1. Thus we get the desired inequality with $a_{\text{pl}} = (6u_0)^{1/k}/2 \in (0, 1)$ as long as K is sufficiently large. \square

5 RSW result

We will prove the following RSW result relating easy crossings to hard crossings of 2×1 rectangles.

Theorem 5.1. *There are constants $\delta_{\text{RSW}} > 0$, $C_{\text{RSW}} < \infty$, $p_{\text{RSW}} > 0$ so that*

$$p_{\text{RSW}} \leq 1/(32 \cdot d_{\text{pass}}^2)^2 \quad (5.1)$$

and, if γ is sufficiently small then the following holds. Let $\mathfrak{R} = [0, S) \times [0, 2S)$. Suppose that, for all $\mathfrak{A} \subseteq \mathfrak{R}$ of aspect ratio between $1/2$ and 2 inclusive, we have

$$\text{CV}^2(\Psi_{\text{easy}}(\mathfrak{A})) < \delta_{\text{RSW}}. \quad (5.2)$$

Then

$$\Theta_{\text{hard}}(\mathfrak{R})[p_{\text{RSW}}] \leq C_{\text{RSW}} \Theta_{\text{easy}}(\mathfrak{R})[p_{\text{RSW}}].$$

Our argument is based on the beautiful proof of the RSW result established for Voronoi percolation in [Tas14]. While our proof has the same structure and uses many of the same geometric constructions, two factors make our setting substantially more complicated than the Voronoi percolation case:

1. We need to take the weights of crossings into account.
2. We do not have as strong a duality theory in the first-passage percolation setting, so rather than comparing crossings for a square and a rectangle, we compare crossings for the easy and hard directions of rectangles.

5.1 Scale and aspect ratio setup

Fix $p_0 \in (0, p_{\text{pl}})$, with p_{pl} as in Proposition 4.13.

We will work with rectangles in the portrait orientation with aspect ratio $\eta = 1 + 2^{-t_0}$, where t_0 is *fixed* but will be chosen later. It will be convenient to work at a series of fixed scales where there are no rounding problems, so for $i \in \mathbb{N}$, let $u_i = [i/2]$, $U_i = 2^{u_i}$, and

$$T_i = 2^{t_0+8} (3/2)^{2[i/2]} U_i = 256 \cdot (3/2)^{2[i/2]} \cdot 2^{t_0+[i/2]}, \quad (5.3)$$

where $[x]$ is the integer part of x and $x = [x] + \{x\}$. (So the sequence $T_0, T_1, T_2, T_3, T_4, T_5, \dots$ is 2^{t_0+7} times the sequence $2, 3, 4, 6, 8, 12, \dots$) In particular, $T_{i+1} \in [4T_i/3, 3T_i/2]$ for each i and $\eta T_i \in \mathbb{N}$ for each i , and if $j \geq i$, we have the simple estimates

$$\sqrt{2}^{j-i-1} T_i \leq T_j \leq \sqrt{2}^{j-i+1} T_i. \quad (5.4)$$

Let $S_i = 2^{s_i} = 2^{t_0+9+\lceil i/2 \rceil}$ be the least dyadic integer greater than or equal to T_i . Let

$$\mathfrak{R}_i = [0, S_i) \times [0, 2S_i), \quad \mathfrak{R}_i^{(\eta)} = [0, T_i) \times [0, \eta T_i),$$

and put

$$w_i^{(\eta)} = \Theta_{\text{easy}}(\mathfrak{R}_i^{(\eta)})[p_0], \quad W_i^{(\eta)} = \max_{j \leq i} w_j^{(\eta)}.$$

It will be convenient to put $w_i^{(\eta)} = W_i^{(\eta)} = 0$ when $i < 0$.

In this section, we will use the notation $\Psi_{X;a}(\mathfrak{R}_i^{(\eta)}; Y)$ for the the minimum Y -weight of a crossing in $\mathfrak{R}_i^{(\eta)}$ joining the four segments $L \times [\eta T_i/2 + a, \eta T_i)$, $L \times [0, \eta T_i/2 - a)$, $R \times [\eta T_i/2 + a, \eta T_i)$, and $R \times [0, \eta T_i/2 - a)$. We also put

$$\begin{aligned} \Psi_{L,a,b}(\mathfrak{B}; Y) &= \Psi_{L;[h/2+a, h/2+b)}(\mathfrak{B}; Y) \\ \Psi_{a,b,R}(\mathfrak{B}; Y) &= \Psi_{[h/2+a, h/2+b), R}(\mathfrak{B}; Y) \end{aligned}$$

where h denotes the height of the box \mathfrak{B} . We moreover extend the π and Θ notation accordingly as in Subsection 2.1. This notation is concordant with the \mathcal{X} and \mathcal{H} notation in [Tas14].

We aim to prove Theorem 5.1, which is about portrait 1×2 rectangles; however, we will argue using rectangles which are portrait but very close to square. In order to conclude, we will need to relate the $w_i^{(\eta)}$ s and the crossing quantiles for 1×2 rectangles. We record the necessary fact in the following lemma, which is simply a translation of Corollary 2.6 into the present notation.

Lemma 5.2. *For any fixed $\eta > 1$ the following holds. There is a constant $C_{\text{stretch}}(\eta) < \infty$ and a probability $p_{\text{stretch}}(\eta) \in (0, 1)$ so that, if γ is sufficiently small, then $w_i^{(\eta)} \leq C_{\text{stretch}}(\eta) \cdot \Theta_{\text{easy}}(\mathfrak{R}_i)[p_{\text{stretch}}(\eta)]$.*

5.2 Gluing

We now begin in earnest the proof of our RSW result.

Lemma 5.3. *There is a $p_1 > 0$, depending only on p_0 , so that the following holds. Let $y \geq w_i^{(\eta)}$, let*

$$\begin{aligned} f_y(\alpha, \beta) &= \mathbf{P}[\Psi_{L,\alpha,\beta}(\mathfrak{R}_i^{(\eta)}) \leq y], \\ g_{w,y}(\alpha) &= f_w(0, \alpha) - f_y(\alpha, \eta T_i/2), \text{ and} \\ \lambda &= \lambda_i^y = \frac{\eta T_i}{8} \wedge \min\{\alpha \in \{1, \dots, \eta T_i\} \mid g_{w_i^{(\eta)}, y}(\alpha) \geq p_0/4\}. \end{aligned} \quad (5.5)$$

Then λ is a well-defined element of $[0, \eta T_i/8]$ and the following two statements both hold:

1. *Either*

- (a) $\mathbf{P}[\Psi_{LR}([0, 2T_i) \times [0, \eta T_i)) \leq 3y] \geq p_1$, or
- (b) $\mathbf{P}[\Psi_{L,\lambda,\eta T_i}(\mathfrak{R}_i^{(\eta)}) \leq y] \geq p_1$.

2. *If $\lambda < \eta T_i/8$, then*

$$\mathbf{P}[\Psi_{L,0,\lambda}(\mathfrak{R}_i^{(\eta)}) \leq w_i^{(\eta)}] \geq \frac{p_0}{4} + \mathbf{P}[\Psi_{L,\lambda,\eta T_i}(\mathfrak{R}_i^{(\eta)}) \leq y].$$

Remark. Note that $f_y(\alpha, \beta)$ is increasing in y , so $g_{w_i, y}(\alpha)$ is decreasing in y and thus λ_i^y is increasing in y . Moreover, for each i , there is a y_i^* so that

$$\lambda_i^{y_i^*} = \eta T_i / 8. \quad (5.6)$$

Proof. First note that g_y is increasing, we have $g_{w_i^{(\eta)}, y}(0) < 0$, and $g_{w_i^{(\eta)}, y}(\eta T_i) = f_{w_i^{(\eta)}}(\eta T_i) \geq p_0/2$. Thus λ is well-defined by the definition in the statement of the theorem. Note that symmetry implies that, for any $\alpha \in (0, \dots, \eta T_i/2)$,

$$p_0/2 \leq f_{w_i^{(\eta)}}(0, \eta T_i/2) \leq f_{w_i^{(\eta)}}(0, \alpha) + f_{w_i^{(\eta)}}(\alpha, \eta T_i/2) \leq f_{w_i^{(\eta)}}(0, \alpha) + f_y(\alpha, \eta T_i/2),$$

so (using (5.5))

$$f_{w_i^{(\eta)}}(0, \lambda) \geq p_0/4 + f_y(\lambda, \eta T_i/2) \geq p_0/4$$

whenever $\lambda < \eta T_i/8$, and

$$f_{w_i^{(\eta)}}(0, \lambda - 1) - f_y(\lambda - 1, \eta T_i/2) < p_0/4. \quad (5.7)$$

In particular, statement 2 holds.

The proof of statement 1 comes down to two cases, depending on $g_{w_i^{(\eta)}, y}(\lambda)$.

Case 1. If $g_{w_i^{(\eta)}, y}(\lambda) \geq 3p_0/8$, then this along with (5.7) implies that

$$\begin{aligned} p_0/8 &< f_{w_i^{(\eta)}}(0, \lambda) - f_y(\lambda, \eta T_i) - [f_{w_i^{(\eta)}}(0, \lambda - 1) - f_y(\lambda - 1, \eta T_i/2)] \\ &\leq f_y(\lambda - 1, \eta T_i/2) - f_y(\lambda, \eta T_i) \leq \mathbf{P}[\Psi_{L, \lambda-1, \lambda}(\mathfrak{R}_i^{(\eta)}) \leq y]. \end{aligned}$$

In words, this means that the probability of a crossing of weight at most y from the left side of $\mathfrak{R}_i^{(\eta)}$ to the point $(T_i, \eta T_i/2 + \lambda - 1)$ is at least $p_0/8$. But then (using (2.2))

$$\begin{aligned} \mathbf{P}[\Psi_{LR}([0, 2T_i] \times [0, \eta T_i]) \leq 3y] &\geq \mathbf{P}[\Psi_{L, \lambda-1, \lambda}([0, T_i] \times [0, \eta T_i]) \leq y] \cdot \mathbf{P}[\Psi_{\lambda-1, \lambda, \mathfrak{R}}([0, T_i] \times [0, \eta T_i]) \leq y] - o(1) \\ &> (p_0/8)^2 - o(1), \end{aligned}$$

so as long as $p_1 \leq (p_0/8)^2$, then statement 1a holds.

Case 2. Now suppose $g_{w_i^{(\eta)}, y}(\lambda) < 3p_0/8$. This means that we have

$$\begin{aligned} p_0/2 &\leq f_{w_i^{(\eta)}}(0, \lambda) + f_{w_i^{(\eta)}}(\lambda, \eta T_i/2) \leq f_{w_i^{(\eta)}}(0, \lambda) + f_y(\lambda, \eta T_i/2) \\ &\leq g_{w_i^{(\eta)}, y}(\lambda) + 2f_y(\lambda, \eta T_i/2) \leq 3p_0/8 + 2f_y(\lambda, \eta T_i/2), \end{aligned}$$

so $f_y(\lambda, \eta T_i/2) \geq p_0/16$. Thus, as long as $p_1 < p_0/16$, statement 1b holds. \square

Lemma 5.4. *If statement 1b of Lemma 5.3 holds, and γ is sufficiently small, then there is a $p_2 > 0$, depending only on p_1 , so that the following holds. Let $y \geq w_i^{(\eta)}$. Suppose that*

$$\eta - \frac{\lambda_i^y}{32T_i} < 1. \quad (5.8)$$

Then if

$$\mu = \mu_i^y \in (\frac{1}{16}\lambda_i^y, \frac{1}{8}\lambda_i^y), \text{ and} \quad (5.9)$$

$$\nu = \nu_i^y = 2\lambda_i^y - \mu_i^y, \quad (5.10)$$

then

$$\mathbf{P}[\Psi_{X; (\nu-\mu)/2}(\mathfrak{R}_i^{(\eta)}) \leq 9y] \geq p_2.$$

¹This case probably never applies if the scale is large enough, but it is easier to tackle it than to prove that it does not occur.

Proof. Note that

$$\{\Psi_{[\eta T_i/2+\nu/2, \eta T_i], [\eta T_i/2+\nu/2, \eta T_i]}(\mathfrak{R}_i^{(\eta)}) \leq 3y\} \supset \{\Psi_{L, \nu/2, \eta T_i/2}(\mathfrak{R}_i^{(\eta)}) \leq y\} \cap \{\Psi_{\nu/2, \eta T_i/2, R}(\mathfrak{R}_i^{(\eta)}) \leq y\}. \quad (5.11)$$

Now we observe that, by (5.11), FKG, (2.2), and Lemma 5.3(1b), we have

$$\begin{aligned} \mathbf{P}[\Psi_{[\eta T_i/2+\nu/2, \eta T_i], [\eta T_i/2+\nu/2, \eta T_i]}(\mathfrak{R}_i^{(\eta)}) \leq 3y] &\geq \mathbf{P}[\Psi_{L, \nu/2, \eta T_i/2}(\mathfrak{R}_i^{(\eta)}) \leq y]^2 - o(1) \\ &\geq \mathbf{P}[\Psi_{L, \lambda, \eta T_i/2}(\mathfrak{R}_i^{(\eta)}) \leq y]^2 - o(1) \geq p_1^2 - o(1). \end{aligned} \quad (5.12)$$

Let $\tilde{\mathfrak{R}}_i^{(\eta)} = [0, T_i) \times [\mu, \eta T_i)$. By (5.8) and (5.9), $\tilde{\mathfrak{R}}_i^{(\eta)}$ is landscape. Let E be the event that there is a left–right path in $\mathfrak{R}_i^{(\eta)}$ connecting the intervals $[\eta T_i/2 + \nu/2, \eta T_i)$ on each side, of weight at most $3y$, that enters the box $\mathfrak{R}_i^{(\eta)} \setminus \tilde{\mathfrak{R}}_i^{(\eta)}$. Then we have that

$$\mathbf{P}[\Psi_{L, \nu/2, \eta T_i/2}(\tilde{\mathfrak{R}}_i^{(\eta)}) \leq 3y] + \mathbf{P}[E] \geq \mathbf{P}[\Psi_{[\eta T_i/2+\nu/2, \eta T_i], [\eta T_i/2+\nu/2, \eta T_i]}(\mathfrak{R}_i^{(\eta)}) \leq 3y] \geq p_1^2 - o(1),$$

where for the second inequality we use (5.12). Thus, either $\mathbf{P}[\Psi_{L, \nu/2, \eta T_i/2}(\tilde{\mathfrak{R}}_i^{(\eta)}) \leq 3y] \geq p_1^2/2 - o(1)$ or $\mathbf{P}[E] \geq p_1^2/2 - o(1)$. We consider each case in turn.

Case 1. First suppose that $\mathbf{P}[E] \geq p_1^2/2$. The intersection of $E_1 = E$ with E_2 , defined to be a copy of E which is vertically flipped and translated upwards by μ , contains $\{\Psi_{X; (\nu-\mu)/2}(\mathfrak{R}_i^{(\eta)}) \leq 3y\}$. This is because the path in E_2 must cross the path from E_1 once on its way from $0 \times [\mu, \mu + \eta T_i/2 - \nu/2)$ to $(\mathfrak{R}_i^{(\eta)} + (0, \mu)) \setminus \mathfrak{R}_i^{(\eta)}$, and another time on its way from $(\mathfrak{R}_i^{(\eta)} + (0, \mu)) \setminus \mathfrak{R}_i^{(\eta)}$ to $\{T_i - 1\} \times [\mu, \mu + \eta T_i/2 - \nu/2)$. (See Figure 5.1a.) Thus we have

$$\mathbf{P}[\Psi_{X; (\nu-\mu)/2}(\mathfrak{R}_i^{(\eta)}) \leq 6y] \geq (p_1^2/2)^2 - o(1)$$

by FKG. This proves the lemma in this case, as long as $p_2 \leq p_1^4/4 - o(1)$.

Case 2. We are left with the case when $\mathbf{P}[\Psi_{L, \nu/2, \eta T_i/2}(\tilde{\mathfrak{R}}_i^{(\eta)}) \leq 3y] \geq p_1^2/2$, which in particular means that

$$\mathbf{P}[\Psi_{LR}(\tilde{\mathfrak{R}}_i^{(\eta)}) \leq 3y] \geq p_1^2/2. \quad (5.13)$$

Observe that the event $\{\Psi_{X; \nu/2}(\mathfrak{R}_i^{(\eta)}) \leq 9y\}$ contains the intersection

$$\{\Psi_{[\eta T_i/2+\nu/2, \eta T_i], [\eta T_i/2+\nu/2, \eta T_i]}(\mathfrak{R}_i^{(\eta)}) \leq 3y\} \cap \{\Psi_{[0, \eta T_i/2-\nu/2), [0, \eta T_i/2-\nu/2)}(\mathfrak{R}_i^{(\eta)}) \leq 3y\} \cap \{\Psi_{BT}(\mathfrak{R}_i^{(\eta)}) \leq 3y\}$$

(see Figure 5.1b), so

$$\begin{aligned} \mathbf{P}[\Psi_{X; \nu/2}(\mathfrak{R}_i^{(\eta)}) \leq 9y] &\geq \mathbf{P}[\Psi_{[\eta T_i/2+\nu/2, \eta T_i], [\eta T_i/2+\nu/2, \eta T_i]}(\mathfrak{R}_i^{(\eta)}) \leq 3y]^2 \mathbf{P}[\Psi_{BT}(\mathfrak{R}_i^{(\eta)}) \leq 3y] \\ &\geq p_1^4 \mathbf{P}[\Psi_{BT}(\mathfrak{R}_i^{(\eta)}) \leq 3y] \end{aligned} \quad (5.14)$$

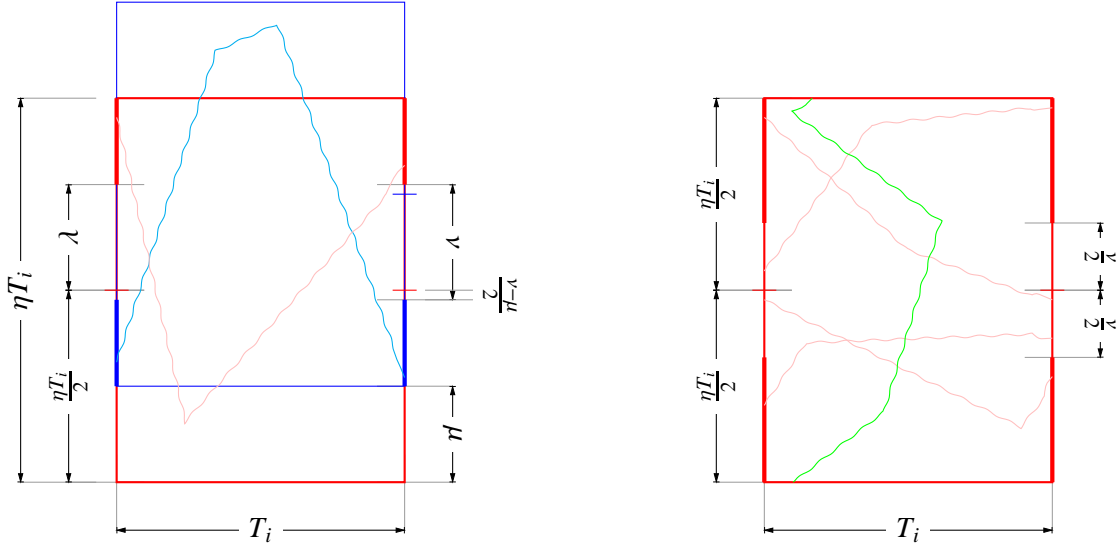
by symmetry, (5.12), and FKG. Now by (5.8) and the definition of μ , we have $\mu/2 \geq \lambda_i^y/32 > (\eta - 1)T_i$, so $2T_i - \eta T_i + \mu > \eta T_i$. Hence, by Corollary 2.5 applied with $k = 2$ (recalling that $\tilde{\mathfrak{R}}_i^{(\eta)}$ is landscape), we have

$$\mathbf{P}[\Psi_{BT}(\mathfrak{R}_i^{(\eta)}) \leq 3y] \geq \mathbf{P}[\Psi_{LR}(\tilde{\mathfrak{R}}_i^{(\eta)}) \leq y]^3 - o(1) \geq p_1^6/8 - o(1),$$

with the second inequality by (5.13). Combining this last inequality with (5.14), we obtain

$$\mathbf{P}[\Psi_{X; (\nu-\mu)/2}(\tilde{\mathfrak{R}}_i^{(\eta)}) \leq 9y] \geq \mathbf{P}[\Psi_{X; \nu/2}(\mathfrak{R}_i^{(\eta)}) \leq 9y] \geq p_1^{10}/8 - o(1),$$

completing the proof of the lemma in this case, as long as $p_2 \leq p_1^{10}/8 - o(1)$. \square



(a) Setting up for the FKG inequality when $\mathbf{P}[E] \geq p_1^2/2$. Combining the entirety of the pink line with the lower part of the blue line gives an “X” shape inside the red $T_i \times \eta T_i$ box with endpoints at least distance $(v-\mu)/2$ from the midline.

(b) Setting up for the FKG inequality when $\mathbf{P}[\Psi_{L,v/2,\eta T_i/2}(\mathfrak{R}_i^{(n)}) \leq y] \geq p_1^2/2$. Combining the green vertical crossing with pieces of the four pink horizontal crossings gives an “X” shape inside the $T_i \times \eta T_i$ box with endpoints at least distance $\lambda \geq (v-\mu)/2$ from the midline.

Figure 5.1: Geometric constructions in the proof of Lemma 5.4.

Lemma 5.5. *There is a p_3 , depending only on p_1 and p_2 , so that if γ is sufficiently small compared to p_1 and p_2 then the following holds. Suppose that $y \geq w_i^{(\eta)}$, $\eta < \frac{256}{255}$, and either*

1. $z \geq w_{i-1}^{(\eta)}$ and $\lambda_i^y \leq \frac{7}{4}\lambda_{i-1}^z$ and $\eta - \frac{\lambda_{i-1}^z}{32T_{i-1}} < 1$ (i.e. (5.8) holds at scale $i-1$ with weight z).
2. $z \geq 0$ and $\lambda_i^y = \eta T_i/8$.

Then

$$\mathbf{P}[\Psi_{\text{LR}}([0, 5T_i/4] \times [0, \eta T_i]) \geq 55y + 8z] \geq p_3.$$

Proof. If

$$\mathbf{P}[\Psi_{\text{LR}}([0, 2T_i] \times [0, \eta T_i]) \leq 3y] \geq p_1,$$

(i.e. if statement 1a from Lemma 5.3 holds) then there is nothing more to show as long as $p_3 \leq p_1 - o(1)$, since horizontally crossing a $T_i \times \eta T_i$ box implies horizontally crossing a $\frac{5}{4}T_i \times \eta T_i$ box. Similarly, if

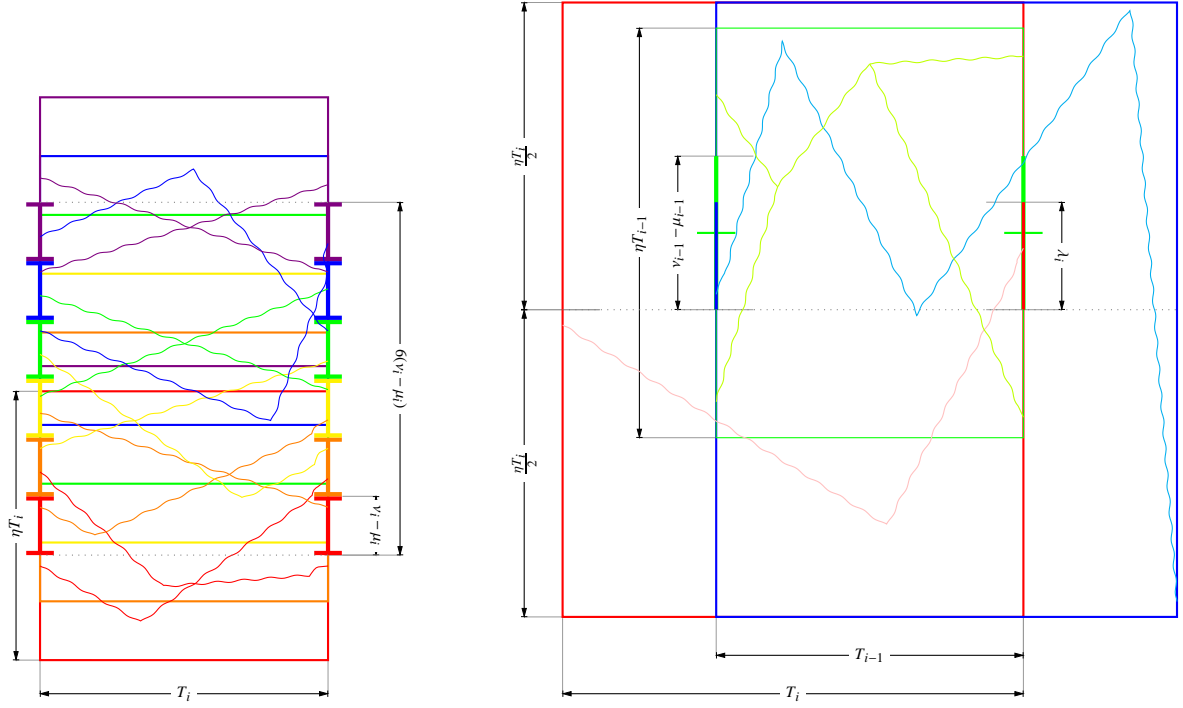
$$\mathbf{P}[\Psi_{\text{LR}}([0, 2T_{i-1}] \times [0, \eta T_{i-1}]) \leq 3z] \geq p_1,$$

then we have (using (5.3))

$$\mathbf{P}[\Psi_{\text{LR}}([0, 5T_i/4] \times [0, \eta T_i]) \leq 4z] \geq \mathbf{P}[\Psi_{\text{LR}}([0, 2T_{i-1}] \times [0, \eta T_{i-1}]) \leq 3z] - o(1) \geq p_1 - o(1),$$

so there is nothing more to show as long as $p_3 \leq p_1 - o(1)$.

Thus from this point on we may assume that statement 1b from Lemma 5.3 holds for both i (with weight $y = y$) and $i-1$ (with weight $y = z$). The rest of the proof is divided into two cases.



(a) A vertical crossing between the two dotted lines is obtained by combining the “X” shapes, which must cross because their endpoints must straddle the interval of their color.

(b) The pink and cyan crossings are guaranteed to be joined by the lime “X” shape, since the pink and cyan crossings must remain within the red and blue boxes and end on the thick red and blue lines, respectively, while the lime “X” must have endpoints off of the thick green lines, which contain the thick red and blue lines.

Figure 5.2: Joining strategies in Lemma 5.5. (Note that we omit the weight superscripts in the dimensions.)

Case 1. If $\lambda_i^y = \eta T_i/8$, then $\eta - \frac{1}{32T_i}\lambda_i^y = \frac{255}{256}\eta < 1$, so Lemma 5.4 implies that

$$\mathbf{P}[\Psi_{X;(\nu_i^y - \mu_i^y)/2}(\mathfrak{R}_i^{(\eta)}) \leq 9y] \geq p_2.$$

Since $\nu_i^y - \mu_i^y = 2(\lambda_i^y - \mu_i^y) \geq \frac{7}{4}\lambda_i^y = \frac{7}{32}\eta T_i$ (recalling (5.9) and (5.10)), the intersection of six vertically-translated copies of $\{\Psi_{X;(\nu_i^y - \mu_i^y)/2}(\mathfrak{R}_i^{(\eta)}) \leq 9y\}$ contains a copy of $\{\Psi_{BT}([0, T_i] \times [0, \frac{21}{16}\eta T_i]) \leq 54y\}$ (as illustrated in Figure 5.2a), hence of $\{\Psi_{BT}([0, T_i] \times [0, \frac{5}{4}T_i]) \leq 54y + 7z\}$. So, by FKG and 2.2,

$$\mathbf{P}[\Psi_{BT}([0, T_i] \times [0, \frac{5}{4}T_i]) \leq 55y + 8z] \geq p_2^6 - o(1).$$

This completes the proof of the lemma in the case when $\lambda_i^y = \eta T_i/8$, as long as $p_3 \leq p_2^6 - o(1)$.

Case 2. Thus we can assume that $\lambda_i^y < \frac{\eta T_i}{8}$, so assumption 1 holds, which is to say that $z \geq w_{i-1}^{(\eta)}$ and $\lambda_i^y \leq \frac{7}{4}\lambda_{i-1}^z$ and (5.8) holds at scale $i-1$ with weight z . Now consider $\mathfrak{R}^1 = \mathfrak{R}_i^{(\eta)}$ and $\mathfrak{R}^2 = \mathfrak{R}_i^{(\eta)} + (T_i - T_{i-1}, 0)$, and

$$\tilde{\mathfrak{R}} = \mathfrak{R}_{i-1}^{(\eta)} + \left(T_i - T_{i-1}, \frac{1}{2}(\eta T_i - \eta T_{i-1} + \nu_{i-1} - \mu_{i-1})\right).$$

Note that, since

$$\frac{\eta T_i - \eta T_{i-1} + \nu_{i-1} - \mu_{i-1}}{2} + \eta T_{i-1} \leq \frac{\eta T_i + \eta T_{i-1}}{2} + \lambda_{i-1} \leq \frac{\eta}{2}(T_i + T_{i-1}) + \frac{\eta}{8}T_{i-1} \leq \frac{\eta}{2}(T_i + \frac{3}{4}T_i) + \frac{3}{32}\eta T_i < \eta T_i,$$

we have $\tilde{\mathfrak{R}} \subset \mathfrak{R}^1 \cap \mathfrak{R}^2$. Since $v_{i-1}^z - \mu_{i-1}^z = 2(\lambda_{i-1}^z - \mu_{i-1}^z) \geq \frac{7}{4}\lambda_{i-1}^z \geq \lambda_i^y$, the event

$$\{\Psi_{X;(v_{i-1}^z - \mu_{i-1}^z)/2}(\tilde{\mathfrak{R}}) \leq 7z\} \cap \{\Psi_{L,0,\lambda_i^y}(\mathfrak{R}^1) \leq y\} \cap \{\Psi_{0,\lambda_i^y,R}(\mathfrak{R}^2) \leq y\}$$

contains, up to course field error (i.e. the error bounded in (2.2)), the event

$$\{\Psi_{LR}([0, 2T_i - T_{i-1}] \times [0, \eta T_i]) \leq 2y + 7z\},$$

since the crossings in the two larger rectangles must both intersect the “X” shape in the smaller rectangle, as they both must end on an interval that is contained in an interval that must be straddled by the endpoints of the “X”. (See Figure 5.2b.) Hence, by FKG and 2.2, we have

$$\begin{aligned} \mathbf{P}[\Psi_{LR}([0, 2T_i - T_{i-1}] \times [0, \eta T_i]) \leq 3y + 8z] &\geq \mathbf{P}[\Psi_{X;(v_{i-1}^z - \mu_{i-1}^z)/2}(\tilde{\mathfrak{R}}) \leq 7z] \cdot \mathbf{P}[\Psi_{L,0,\lambda_i^y}(\mathfrak{R}^1) \leq y]^2 - o(1) \\ &\geq p_1^2 \mathbf{P}[\Psi_{X;(v_{i-1}^z - \mu_{i-1}^z)/2}(\tilde{\mathfrak{R}}) \leq 7z] - o(1). \end{aligned}$$

Now by Lemma 5.4, recalling our assumption that (5.8) holds at scale $i-1$ with weight z , if γ is sufficiently small compared to p_2 , we have

$$\mathbf{P}[\Psi_{X;(v_{i-1}^z - \mu_{i-1}^z)/2}(\tilde{\mathfrak{R}}) \leq 9z] \geq p_2.$$

So

$$\mathbf{P}[\Psi_{LR}([0, 5T_i/4] \times [0, \eta T_i]) \leq 4y + 9z] \geq \mathbf{P}[\Psi_{LR}([0, 2T_i - T_{i-1}] \times [0, \eta T_i]) \leq 3y + 8z] - o(1) \geq p_1^2 p_2 / 2 - o(1),$$

completing the proof of the lemma in the case when $\lambda_i^y < \eta T_i / 8$, as long as $p_3 \leq \frac{1}{2} p_1^2 p_2 - o(1)$. \square

Lemma 5.6. *There are constants c_1 and p_4 , depending only on p_3 , so that the following holds. Let $j \geq i + 8$. Suppose that $\eta \leq 9/8$, $\lambda_j^{w_j^{(\eta)}} \leq \eta T_i$, γ is sufficiently small compared to p_3 , and*

$$\mathbf{P}[\Psi_{LR}([0, 5T_i/4] \times [0, \eta T_i]) \leq y] \geq p_3. \quad (5.15)$$

Then

$$\mathbf{P}[\Psi_{LR}([0, 2T_j] \times [0, \eta T_j]) \leq 2w_j^{(\eta)} + c_1 y] \geq p_4 - o_{j \rightarrow i}(1).$$

Proof. Since $j \geq i + 8$, we have $T_j \geq 16T_i$, so $\lambda_j^y \leq \eta T_i < \eta T_j / 8$, so by statement 2 of Lemma 5.3, we have

$$\mathbf{P} \left[\Psi_{L,0,\lambda_j^{w_j^{(\eta)}}}(\mathfrak{R}_j^{(\eta)}) \leq w_i^{(\eta)} \right] \geq p_0/4 \quad (5.16)$$

and

$$\mathbf{P} \left[\Psi_{0,\lambda_j^{w_j^{(\eta)}},R}(\mathfrak{R}_j^{(\eta)} + (T_j, 0)) \leq w_i^{(\eta)} \right] \geq p_0/4. \quad (5.17)$$

Now we can build an annulus \mathfrak{Q} —whose inner square has width ηT_i and whose outer square has width $3\eta T_i$ —inside $\mathfrak{R}_j^{(\eta)} \cup (\mathfrak{R}_j^{(\eta)} + (T_j, 0))$, surrounding $T_j \times \left[\eta T_j / 2, \eta T_j / 2 + \lambda_j^{w_j^{(\eta)}} \right]$.

By (5.15), our upper bound on η , and Corollary 2.5, we have constants c_2 and p_5 , depending only on p_3 and the facts that η is a constant amount less than $5/4$ and that γ is sufficiently small, so that

$$\mathbf{P}[\Psi_{LR}([0, 3\eta T_i] \times [0, \eta T_i]) \leq c_2 y] \geq p_5.$$

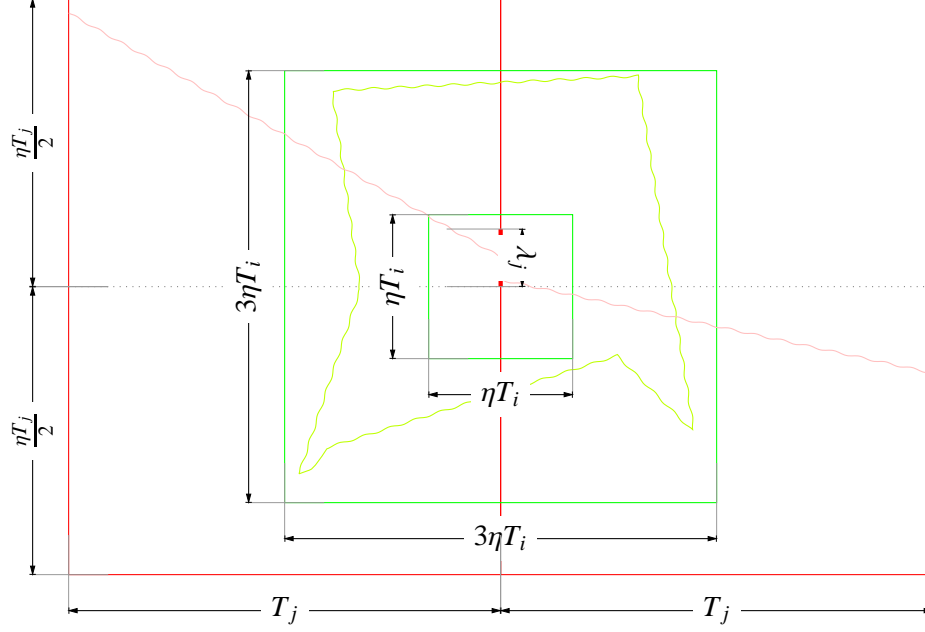


Figure 5.3: Geometric construction in the proof of Lemma 5.6. The two pink crossings are connected by the lime circuit. (Again we omit the weight subscript.)

Let E denote the event that there is a circuit of $Y_{\mathfrak{R}_j^{(\eta)} \cup (\mathfrak{R}_j^{(\eta)} + (T_j, 0))}$ -weight at most $5c_2y$ around \mathfrak{Q} , and let E_1, E_2, E_3, E_4 denote rotated and translated copies of $\{\Psi_{\text{LR}}([0, 3\eta T_i] \times [0, \eta T_i]) \leq c_2y\}$ whose intersection, up to course field error, contains E . Now $\mathbf{P}[E_a] = \mathbf{P}[\Psi_{\text{LR}}([0, 3\eta T_i] \times [0, \eta T_i]) \leq c_2y] \geq p_5$, and so, by FKG and 2.2, we have

$$\mathbf{P}[E] \geq p_5^4 - o_{j-i}(1). \quad (5.18)$$

Since, up to course field error, we have

$$\{\Psi_{\text{LR}}([0, 2T_j] \times [0, \eta T_j]) \leq 2w_j^{(\eta)} + 5c_2y\} \supset E \cap \{\Psi_{\text{L}, 0, \lambda_j^{(\eta)}}(\mathfrak{R}_j^{(\eta)}) \geq w_i^{(\eta)}\} \cap \{\Psi_{\text{R}, 0, \lambda_j^{(\eta)}}(\mathfrak{R}_j^{(\eta)} + (T_j, 0)) \leq w_i^{(\eta)}\}$$

(see Figure 5.3), the FKG inequality, along with (5.16), (5.17), and (5.18), tells us that

$$\mathbf{P}[\Psi_{\text{LR}}([0, 2T_j] \times [0, \eta T_j]) \leq 3w_j^{(\eta)} + 5c_2y] \geq (p_0/4)^2 (p_5/2)^4 - o_{j-i}(1),$$

establishing the lemma with $c_1 = 5c_2$ and $p_4 = (p_0/4)^2 (p_5/2)^4$. \square

5.3 Multiscale analysis

Lemma 5.7. *Let c_3 be so large that*

$$(1 - p_4^{15})^{\lfloor c_3/4 \rfloor} \leq p_0/8. \quad (5.19)$$

Suppose that $\eta \leq \frac{256}{255}$ and that (5.15) holds for i and y . For any $\Delta \geq 6$, there is a $j \in [i + \Delta, i + \Delta + c_3]$ so that, if γ is sufficiently small relative to Δ , then

$$\lambda_j^{21w_j^{(\eta)} + 10c_1y} \geq \eta T_i. \quad (5.20)$$

Proof. Let $\tilde{j} = i + \Delta + c_3$. Suppose for the sake of contradiction that, for all $i + \Delta < j \leq \tilde{j}$, we have $\lambda_j^{w_j^{(\eta)}} < \eta T_i$, and moreover that we have

$$\lambda_j^{w_j^{(\eta)} + 2(10W_{\tilde{j}} + 5c_1y)} < \eta T_i. \quad (5.21)$$

Then Lemma 5.6 implies that

$$\mathbf{P}[\Psi_{\text{LR}}([0, 2T_j] \times [0, \eta T_j])] \leq 2w_j^{(\eta)} + c_1y \geq p_4 - o_\Delta(1)$$

for each $i + \Delta < j \leq \tilde{j}$. By Corollary 2.5, this yields

$$\mathbf{P}[\Psi_{\text{LR}}([0, 3\eta T_j] \times [0, \eta T_j])] \leq 10w_j^{(\eta)} + 5c_1y \geq p_4^5 - o_\Delta(1). \quad (5.22)$$

Let

$$\begin{aligned} J_1 &= \{\Psi_{\text{L}, 0, \eta T_i}(\mathfrak{R}_{\tilde{j}}^{(\eta)}) \leq w_{\tilde{j}}^{(\eta)}\} \text{ and} \\ J_2 &= \{\Psi_{\text{L}, \eta T_i, \eta T_{\tilde{j}}/2}(\mathfrak{R}_{\tilde{j}}^{(\eta)}) \leq w_{\tilde{j}}^{(\eta)} + 2(10W_{\tilde{j}} + 5c_1y)\}. \end{aligned}$$

Then (5.21) and Lemma 5.3(2) imply that we have

$$\mathbf{P}[J_1] - \mathbf{P}[J_2] \geq p_0/4. \quad (5.23)$$

Let E be the event that there is a path in $\mathfrak{R}_{\tilde{j}}^{(\eta)}$ of weight at most $2(10W_{\tilde{j}} + 5c_1y)$, from $\{T_{\tilde{j}} - 1\} \times [\eta T_{\tilde{j}}/2 + \eta T_i, \eta T_{\tilde{j}}]$ to $\{T_{\tilde{j}} - 1\} \times [0, \eta T_{\tilde{j}}/2]$. Note that $J_1 \cap E \subset J_2$, so

$$\mathbf{P}[J_2] \geq \mathbf{P}[J_1 \cap E] \geq \mathbf{P}[J_1]\mathbf{P}[E] \quad (5.24)$$

by the FKG inequality. Combining (5.23) and (5.24), we get that

$$\mathbf{P}[E^c] \geq \mathbf{P}[J_1]\mathbf{P}[E^c] \geq p_0/4. \quad (5.25)$$

For each $i + \Delta \leq j < \tilde{j}$ such that $j \in 4\mathbf{Z}$, let E_1^j, E_2^j, E_3^j be the events that there are hard crossings—of weight at most $10w_j^{(\eta)} + 5c_1y$ —in respectively, three rectangles of shorter side-length ηT_j and longer side-length $3\eta T_j$, that together form a “C” shape connecting $\{T_j - 1\} \times [\eta T_j/2 + \eta T_i, \eta T_j]$ to $\{T_j - 1\} \times [0, \eta T_j/2]$, and which moreover are chosen so that the blow-ups of the rectangles only intersect other rectangles corresponding to the same j . The setup is illustrated in Figure 5.4.

By (5.22) we have

$$\mathbf{P}[E_\alpha^j] \geq p_4^5 - o_\Delta(1).$$

Let $\tilde{E}_1^j, \tilde{E}_2^j, \tilde{E}_3^j$ be defined in the same way as E_1^j, E_2^j, E_3^j , except with the requirement that the $Y_{\mathfrak{R}_{\tilde{j}}^{(\eta)}}$ -weight of the paths be at most $2(10w_j^{(\eta)} + 5c_1y)$, rather than that the weight of the paths with respect to the GFF in their own rectangles be at most $10w_j^{(\eta)} + 5c_1y$.

For each j , we have that $\tilde{E}_1^j \cap \tilde{E}_2^j \cap \tilde{E}_3^j \subset E$. Let $Z = \max_{\alpha, j, x} W_\alpha^j(x)$, where W_α^j is the course field correction term for the rectangle in E_α^j as in 2.2. Now note that $\tilde{E}_1^j \cap \tilde{E}_2^j \cap \tilde{E}_3^j \supset E_1^j \cap E_2^j \cap E_3^j \cap \{Z \leq 2\}$, so we can compute

$$\begin{aligned} \mathbf{P}[E^c] &\leq \mathbf{P}\left[\bigcap_{\substack{i+\Delta \leq j < \tilde{j} \\ j \in 4\mathbf{Z}}} (\tilde{E}_1^j \cap \tilde{E}_2^j \cap \tilde{E}_3^j)^c\right] \leq \mathbf{P}\left[\bigcap_{\substack{i+\Delta \leq j < \tilde{j} \\ j \in 4\mathbf{Z}}} ((E_1^j \cap E_2^j \cap E_3^j)^c \cup \{Z > 2\})\right] \\ &\leq o_\Delta(1) + \prod_{\substack{i+\Delta \leq j < \tilde{j} \\ j \in 4\mathbf{Z}}} (1 - \mathbf{P}[E_1^j \cap E_2^j \cap E_3^j]) \leq \prod_{\substack{i+\Delta \leq j < \tilde{j} \\ j \in 4\mathbf{Z}}} (1 - \mathbf{P}[E_1^j]^3) + o_\Delta(1) \leq (1 - p_4^{15})^{\lfloor c_3/4 \rfloor} + o_\Delta(1). \end{aligned}$$

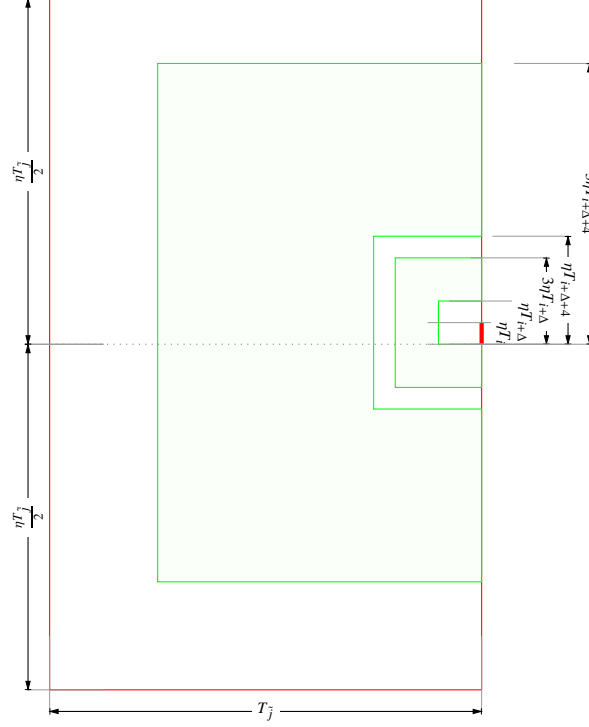


Figure 5.4: Geometric construction in the proof of Lemma 5.7. (In reality there would be *many* more half-annuli!)

Now if γ is small enough (relative to Δ), then combined with (5.19) this implies that $\mathbf{P}[E^c] < p_0/4$, contradicting (5.25). So either, for some $i + \Delta < j \leq \tilde{j}$, we have $\lambda_j^{w_j^{(\eta)}} \geq \eta T_i$, or else we have $\lambda_j^{w_j^{(\eta)} + 2(10W_j^{(\eta)} + 5c_1y)} \geq \eta T_i$, implying (5.20) in any case. \square

Lemma 5.8. Write $f(k) = \lambda_k^{y_k}$ for some sequence $y_k \geq w_k^{(\eta)}$. Suppose that $f(k_0) \geq a\eta T_{k_0}$. Then if c is so large that $\left(\frac{7}{4}\right)^c \frac{1}{\sqrt{2}^{c-1}} > \frac{1}{a}$, then there is a $k \in (k_0, k_0 + c]$ so that $f(k) \leq \frac{7}{4}f(k-1)$ and $f(k-1) \geq a\eta T_{k-1}$.

Proof. If $f(k) \geq \frac{7}{4}f(k-1)$ for all $k_0 < k \leq k_0 + c$, then we have (using (5.4))

$$\frac{1}{2}\eta T_{k_0+c} \geq f(k_0+c) \geq \left(\frac{7}{4}\right)^c a\eta T_{k_0} \geq \left(\frac{7}{4}\right)^c a\eta \frac{1}{\sqrt{2}^{c+1}} T_{k_0+c},$$

contradicting our assumption on c . Therefore, there is some $k \in (k_0, k_0 + c]$ so that $f(k) \leq \frac{7}{4}f(k-1)$. Moreover, if we choose the *first* such k , then we have

$$\frac{f(j)}{T_j} \geq \frac{7}{4} \cdot \frac{f(j-1)}{T_j} \geq \frac{7}{4} \cdot \frac{f(j-1)}{3T_{j-1}/2} \geq \frac{f(j-1)}{T_{j-1}}$$

for all $k_0 < j < k$, so by induction we have $f(k-1) \geq a\eta T_{k-1}$. \square

Lemma 5.9. Let $c_5 = \max\{1323, 630c_1\}$. Fix $\Delta \geq 6$ and suppose that

$$1 < \eta \leq 1 + \frac{1}{32\sqrt{2}^{\Delta+c_3+1}}. \quad (5.26)$$

Then if γ is sufficiently small relative to Δ , then there is a $\chi(\Delta) \geq \Delta$ so that if (5.15) holds for i and y , then there is a $k \in [i + \Delta, i + \chi(\Delta)]$ so that (5.15) holds for $i = k$ and $y = c_5(W_k^{(\eta)} + y)$.

Proof. By Lemma 5.7, there is a $j \in [i + \Delta, i + \Delta + c_3]$ so that (using (5.4)) we have $\lambda_j^{21W_j^{(\eta)} + 10c_1y} \geq \eta T_i \geq \frac{\eta T_j}{\sqrt{2}^{\Delta+c_3+1}}$. Let $\xi(\Delta)$ be so large that $\frac{(7/4)^{\xi(\Delta)}}{\sqrt{2}^{\xi(\Delta)-1}} > \sqrt{2}^{\Delta+c_3+1}$. Then if we put $f(k) = \lambda_k^{11eW_k^{(\eta)} + 5ec_1y}$ and let $\chi(\Delta) = \xi(\Delta) + c_3$, by Lemma 5.8 there is some $k \in (j, j + \xi(\Delta)] \subset [i + \Delta, i + \Delta + c_3 + \xi(\Delta)] = [i + \Delta, i + \chi(\Delta)]$ so that

$$\lambda_k^{21W_k^{(\eta)} + 10c_1y} \leq \frac{7}{4} \lambda_{k-1}^{21W_{k-1}^{(\eta)} + 10c_1y}$$

and

$$\frac{1}{32} \lambda_{k-1}^{21W_{k-1}^{(\eta)} + 10c_1y} \geq \frac{\eta T_{k-1}}{32 \sqrt{2}^{\Delta+c_3+1}} \geq \frac{T_{k-1}}{32 \sqrt{2}^{\Delta+c_3+1}} > (\eta - 1) T_{k-1},$$

with the last inequality by (5.26). Thus the hypotheses of Lemma 5.5 hold with $i = k$, $y = 21W_k^{(\eta)} + 10c_1y$, and $z = 21W_{k-1}^{(\eta)} + 10c_1y$ (where the left-hand sides are in the notation of the statement of Lemma 5.5 and the right-hand sides are in the notation of the present proof). This means that

$$\mathbf{P}[\Psi_{\text{LR}}([0, 5T_k/4] \times [0, \eta T_k]) \geq 55(21W_k^{(\eta)} + 10c_1y) + 8(21W_{k-1}^{(\eta)} + 10c_1y)] \geq p_3,$$

which is to say that (5.15) holds with $y = c_5(W_k^{(\eta)} + y)$ (where again the left-hand side is in the notation of (5.15) and the right-hand side is in the present notation). \square

Lemma 5.10. Fix $\Delta \geq 6$ and suppose that $\eta - 1 \leq 2^{-(\Delta+c_3+7)}$. Then there is an increasing sequence $1 = i_1, i_2, i_3, \dots$ so that

$$i_{r+1} \in [i_r + \Delta, i_r + \chi(\Delta)], \quad (5.27)$$

and, for each r , (5.15) holds for $i = i_r$ and

$$y = \sum_{s=1}^r c_5^{r+1-s} (W_{i_s}^{(\eta)} \vee y_1^*). \quad (5.28)$$

Proof. According to (5.6), we have $\lambda_1^{y_1^*} = \eta T_1/8$. This means that Lemma 5.5 applies, so (5.15) holds for $i = 1$ and $y = 42y_1^*$. In other words, if we put $i_1 = 1$, then the conclusion of the lemma holds for $r = 1$.

Now we claim that once we have chosen a suitable i_r , then we can also choose a suitable i_{r+1} . Indeed, if (5.15) holds for $i = i_r$ and

$$y = \sum_{s=1}^r c_5^{r+1-s} (W_{i_s}^{(\eta)} \vee y_1^*),$$

then Lemma 5.9 implies that there is an $i_{r+1} \in [i_r + \Delta, i_r + \chi(\Delta)]$ so that (5.15) holds for $i = i_{r+1}$ and

$$y = c_5 \left(W_{i_{r+1}} + \sum_{s=1}^r c_5^{r+1-s} (W_{i_s}^{(\eta)} \vee y_1^*) \right) \leq \sum_{s=1}^{r+1} c_5^{r+2-s} (W_{i_s}^{(\eta)} \vee y_1^*),$$

hence for $y = \sum_{s=1}^{r+1} c_5^{r+2-s} (W_{i_s}^{(\eta)} \vee y_1^*)$ as well. This finishes the inductive step of the proof of the lemma. \square

The next lemma uses the fact that our desired results at a given scale imply the same results at constant multiples of the scale to extend Lemma 5.10 to all scales, and also to better-shaped boxes.

Lemma 5.11. Fix $\Delta \geq 6$ and suppose that $\eta - 1 \leq 2^{-(\Delta+c_3+7)}$. We have constants $p(\Delta)$ and $C(\Delta)$ so that for each $i \geq 1$, we have

$$\Theta_{\text{hard}}(\mathcal{R}_i)[p(\Delta)] \leq C(\Delta) \sum_{j=0}^{\lfloor i/\Delta \rfloor} c_5^j (W_{i-1-j\Delta}^{(\eta)} \vee y_1^*).$$

Proof. By Lemma 5.10, there is an i_r so that $i - 1 - \chi(\Delta) \leq i_r \leq i - 1$ and

$$\mathbf{P}[\Psi_{\text{LR}}([0, 5T_{i_r}/4) \times [0, \eta T_{i_r})) \leq y_r] \leq p_3, \quad (5.29)$$

where

$$y_r = \sum_{\alpha=1}^r c_5^{r+1-\alpha} (W_{i_\alpha}^{(\eta)} \vee y_1^*).$$

Note that (5.27) implies that, for each α , we have $i_\alpha \leq i_r - (r - \alpha)\Delta$. This means that

$$y_r \leq \sum_{\alpha=1}^r c_5^{r+1-\alpha} (W_{i_r-(r-\alpha)\Delta}^{(\eta)} \vee y_1^*) = \sum_{j=0}^{r-1} c_5^{j+1} (W_{i_r-j\Delta}^{(\eta)} \vee y_1^*) \leq \sum_{j=0}^{r-1} c_5^{j+1} (W_{i-1-j\Delta}^{(\eta)} \vee y_1^*).$$

Now Corollary 2.5 and (2.2) imply the desired result. \square

We are finally ready to prove our RSW result.

Proof of Theorem 5.1. Fix $K = 2^k \geq K_{\text{pl}}$, and choose κ so large that

$$a_{\text{pl}}^\kappa < \frac{1}{4c_5} \text{ and} \quad (5.30)$$

$$c_5^{\frac{1}{2\kappa}} < 1/a_{\text{pl}} \quad (5.31)$$

Put $\Delta = \lceil 2\kappa \rceil$ and apply Lemma 5.11. Fix η as in the statement of that lemma; then we have

$$\begin{aligned} \Theta_{\text{hard}}(\mathfrak{R}_i)[p(\Delta)] &\leq C(\Delta) \sum_{j=0}^{\lfloor \frac{i}{2\kappa} \rfloor} c_5^j (W_{i-1-j\Delta}^{(\eta)} \vee y_1^*) \leq C_{\text{stretch}}(\eta) C(\Delta) \sum_{j=0}^{\lfloor \frac{i}{2\kappa} \rfloor} c_5^j \left(\max_{\alpha \leq i-1-j\Delta} \Theta_{\text{easy}}(\mathfrak{R}_\alpha)[p_{\text{stretch}}(\eta)] \vee y_1^* \right) \\ &\leq C_{\text{stretch}}(\eta) C(\Delta) \sum_{j=0}^{\lfloor \frac{i}{2\kappa} \rfloor} c_5^j \max_{\alpha \leq i-1-j\Delta} \Theta_{\text{easy}}(\mathfrak{R}_\alpha)[p_{\text{stretch}}(\eta)] + C_3 y_1^* \sum_{j=0}^{\lfloor \frac{i}{2\kappa} \rfloor} c_5^j, \end{aligned} \quad (5.32)$$

with the second inequality by Lemma 5.2.

Our goal is to relate the sums in (5.32) to a quantile of an easy crossing of \mathfrak{R}_i , and our primary tool will be the *a priori* power-law lower bound of sufficiently small crossing quantiles given in Proposition 4.13. However, Proposition 4.13 only relates very small quantiles, and the quantiles in (5.32) (coming from Lemma 5.2) are very large. This is the reason for the assumption (5.2): by applying (3.2), this assumption lets us relate very small and very large quantiles, assuming δ is chosen sufficiently small.

Let's put this plan into action. For each j , we have

$$\max_{\alpha \leq i-1-j\Delta} \Theta_{\text{easy}}(\mathfrak{R}_\alpha)[p_{\text{stretch}}(\eta)] \leq C \max_{\alpha \leq i-1-j\Delta} \Theta_{\text{easy}}(\mathfrak{R}_\alpha)[p_{\text{pl}}]$$

(with p_{pl} as in Proposition 4.13) by (5.2) and (3.2), choosing δ small enough (depending on p_{stretch} and p_{pl}) so that the necessary assumptions hold. But then by Proposition 4.13, we have

$$\max_{\alpha \leq i-1-j\Delta} \Theta_{\text{easy}}(\mathfrak{R}_\alpha)[p_{\text{stretch}}(\eta)] \leq CC_{\text{pl}} a_{\text{pl}}^{j\Delta+1} \cdot \Theta_{\text{easy}}(\mathfrak{R}_i)[q_{\text{pl}}].$$

This gives us

$$\sum_{j=0}^{\lfloor \frac{i}{2\kappa} \rfloor} c_5^j \max_{\alpha \leq i-1-j\Delta} \Theta_{\text{easy}}(\mathfrak{R}_\alpha)[p_{\text{stretch}}(\eta)] \leq CC_{\text{pl}} \Theta_{\text{easy}}(\mathfrak{R}_i)[q] \sum_{j=0}^{\lfloor \frac{i}{2\kappa} \rfloor} c_5^j a_{\text{pl}}^{j\Delta+1} \leq C' \Theta_{\text{easy}}(\mathfrak{R}_i)[q], \quad (5.33)$$

where in the last inequality we use (5.30). Moreover, we have

$$\sum_{j=0}^{\lfloor \frac{i}{2k} \rfloor} c_5^j \leq \frac{c_5^{\frac{i}{2k}+1} - 1}{c_5 - 1} \leq C'' \Theta_{\text{easy}}(\mathfrak{R}_i)[q], \quad (5.34)$$

with the last inequality by (5.31) and Proposition 4.13. Plugging (5.33) and (5.34) into (5.32), we have in conclusion, choosing $p_{\text{RSW}} \leq \min\{p(\Delta), p_{\text{pl}}, (32 \cdot d_{\text{pass}}^2)^{-2}\}$, that (5.1) holds and

$$\Theta_{\text{hard}}(\mathfrak{R}_S)[p_{\text{RSW}}] \leq \Theta_{\text{hard}}(\mathfrak{R}_S)[p(\Delta)] \leq C''' \Theta_{\text{easy}}(\mathfrak{R}_S)[q] \leq C''' \Theta_{\text{easy}}(\mathfrak{R}_S)[p_{\text{RSW}}]$$

with the second inequality by (3.2) and (5.2) as long as δ is sufficiently small compared to p_{RSW} . \square

6 Upper bounds on FPP weight and geodesic length

In this section we derive upper bounds on the crossing weight, geodesic length, and box diameter.

6.1 Crossing weight upper bound

We want to derive a right-tail bound on the crossing weight in terms of the hard crossing weights at a smaller scale. We show this by showing that hard crossings from smaller scales can be glued together to get a crossing at a larger scale, and that there are many nearly-independent opportunities for this to happen, so we get good control on the right tail of the crossing weight.

Let $\mathfrak{R} = [0, KS) \times [0, LS)$ with $K = 2^k$ and $L = 2^l$. For convenience assume that $3 \mid L$. Let $\mathfrak{C} = [0, S)^2$, $\mathfrak{A} = [0, S) \times [0, 2S)$ and $\mathfrak{A}' = [0, 2S) \times [0, S)$. Index the dyadic subboxes of \mathfrak{B} having side length S by row and column according to the following layout:

$$\begin{array}{ccc} \mathfrak{C}_{11} & \cdots & \mathfrak{C}_{1L} \\ \vdots & \ddots & \vdots \\ \mathfrak{C}_{L1} & \cdots & \mathfrak{C}_{LK} \end{array}$$

Proposition 6.1. *If $u > 0$, we have*

$$\mathbf{P}[\Psi_{\text{LR}}(\mathfrak{R}) \geq 2uKE\Psi_{\text{hard}}(\mathfrak{A})] \leq u^{-L/3} + o_{K,L}(1) \quad (6.1)$$

Moreover, if $u \geq u_0$ (defined in Lemma 2.2), then we have

$$\mathbf{P}[\Psi_{\text{LR}}(\mathfrak{R}) \geq 2uKE\Psi_{\text{hard}}(\mathfrak{A})] \leq u^{-L/4} + \exp\left(-\Omega(1) \cdot \frac{(\log u)^2}{\log(K \vee L)}\right). \quad (6.2)$$

Proof. For each $0 \leq j \leq L-1$ such that $3 \mid j$, let $\overline{\Psi}_j = \Psi_{\text{LR}}([0, jS) + [0, KS) \times [0, S))$. Then for each j , by (2.2) and the strategy illustrated in Figure 2.1 we have

$$\Psi_{\text{LR}}(\mathfrak{R}) \leq (1 + o_{K,L}(1)) \overline{\Psi}_j \leq (1 + o_{K,L}(1)) \cdot \left[\sum_{i=1}^{K-1} \Psi_{\text{hard}}(\mathfrak{C}_{1,i} \cup \mathfrak{C}_{1,i+1}) + \sum_{i=2}^{K-1} \Psi_{\text{hard}}(\mathfrak{C}_{1,i} \cup \mathfrak{C}_{2,i}) \right]$$

in probability. Thus we have

$$\mathbf{E} \overline{\Psi}_j \leq (1 + o_{K,L}(1))(2K-3) \mathbf{E} \Psi_{\text{hard}}(\mathfrak{A}) \leq 2KE\Psi_{\text{hard}}(\mathfrak{A}) \quad (6.3)$$

as long as γ is sufficiently small compared to K . Applying Markov's inequality gives us

$$\mathbf{P}[\overline{\Psi}_j \geq 2uKE\Psi_{\text{hard}}(\mathfrak{A})] \leq 1/u.$$

Since $\overline{\Psi}_0, \overline{\Psi}_3, \dots, \overline{\Psi}_{L-1}$ are independent, we have (6.1) by (2.2) and (6.2) by (2.3) and the assumption that $u \geq u_0$. \square

Corollary 6.2. *If γ is sufficiently small, then there are constants $b_{\text{pl}}, C'_{\text{pl}} < \infty$ so that for any K and S we have*

$$\mathbf{E}\Psi_{\text{hard}}([0, 2^r S) \times [0, 2^{r+1} S)) \leq C'_{\text{pl}} b_{\text{pl}} \mathbf{E}\Psi_{\text{hard}}([0, S) \times [0, 2S)).$$

Moreover, b_{pl} can be made arbitrarily close to 1 by making γ sufficiently small.

Proof. By (6.3), in the notation of Proposition 6.1 we have $\mathbf{E}\Psi_{\text{hard}}(\mathfrak{R}) \leq (2 + o_{K,L}(1))K\mathbf{E}\Psi_{\text{hard}}(\mathfrak{U})$. The statement then follows by induction on the scale after choosing K, L sufficiently large and γ sufficiently small. \square

6.2 Expected crossing length upper bound

Let $\mathfrak{R} = [0, KS) \times [0, LS)$ with $K = 2^k$ and $L = 2^l$. Let $\mathfrak{U} = [0, S) \times [0, 2S)$ and $\mathfrak{U}' = [0, 2S) \times [0, S)$. We want to show that a left–right crossing of \mathfrak{R} will typically not enter too many dyadic $S \times S$ subboxes of \mathfrak{R} . Our strategy will be to show that a path that enters many boxes will likely have a higher weight than the tail-bound value from Proposition 6.1.

Proposition 6.3. *For any u and p , we have*

$$\mathbf{P}\left[M_{\text{LR};S}(\mathfrak{R}) \geq K \max\left\{1, 4uu_0 C_{\text{PD}} \frac{\mathbf{E}\Psi_{\text{hard}}(\mathfrak{U})}{\Theta_{\text{easy}}(\mathfrak{U})[p]}\right\}\right] \leq u^{-L/3} + 2L(2d_{\text{pass}}^2 \sqrt{p})^K + o_{K,L}(1).$$

Proof. According to Proposition 6.1, with probability at least $1 - u^{-L/3} - o_{K,L}(1)$, we have

$$\Psi_{\text{LR}}(\mathfrak{R}) \leq 2uK\mathbf{E}\Psi_{\text{hard}}(\mathfrak{U}).$$

On the other hand, by Proposition 4.11 and Proposition 4.5, with probability at least $1 - 2L(2d_{\text{pass}}^2 \sqrt{p})^N - o_{K,L}(1)$ we have

$$\min_{\|\pi\|_S \geq C_{\text{PD}} N} \psi(\pi; Y_{\mathfrak{R}}) > \frac{N}{2u_0} \Theta_{\text{easy}}(\mathfrak{U})[p].$$

Thus if

$$\frac{N}{2u_0} \Theta_{\text{easy}}(\mathfrak{U})[p] \geq 2uK\mathbf{E}\Psi_{\text{hard}}(\mathfrak{U}),$$

then with probability at least $1 - u^{-L/3} - 2L(2d_{\text{pass}}^2 \sqrt{p})^N - o_{K,L}(1)$, we have $M_{\text{LR};S}(\mathfrak{R}) \leq C_{\text{PD}} N$. Putting

$$N = K \max\left\{1, 4u_0 u \frac{\mathbf{E}\Psi_{\text{hard}}(\mathfrak{U})}{\Theta_{\text{easy}}(\mathfrak{U})[p]}\right\}$$

yields the desired result. \square

Proposition 6.4. *There is a $\delta_0 > 0$ and a $C_{\text{CL}} > 0$ so that the following holds. If $\text{CV}^2(\Psi_{\text{easy}}(\mathfrak{E})) < \delta_{\text{RSW}}$ whenever $\mathfrak{E} \subseteq [0, S) \times [0, 2S)$ has aspect ratio between 1/2 and 2 inclusive, and $\text{CV}^2(\Psi_{\text{hard}}(\mathfrak{U})) < \delta < \delta_0$, then we have*

$$\mathbf{E}M_{\text{LR};S}(\mathfrak{R}) \leq K\left(C_{\text{CL}} + K\left[2^{-L/3} + 2L(2d_{\text{pass}}^2 \sqrt{p_{\text{RSW}}})^K\right]\right) + o_{K,L}(1).$$

Remark. Note that (5.1) implies that the third term decays geometrically as $K \rightarrow \infty$.

Proof. Putting $p = p_{\text{RSW}}$ in the previous lemma, we have

$$\mathbf{E}M_{\text{LR};S}(\mathfrak{R}) \leq K \max\left\{1, 4u_0 u C_{\text{PD}} \frac{\mathbf{E}\Psi_{\text{hard}}(\mathfrak{U})}{\Theta_{\text{easy}}(\mathfrak{U})[p]}\right\} + K^2\left[u^{-L/3} + 2L(2d_{\text{pass}}^2 \sqrt{p})^K\right] + o_{K,L}(1).$$

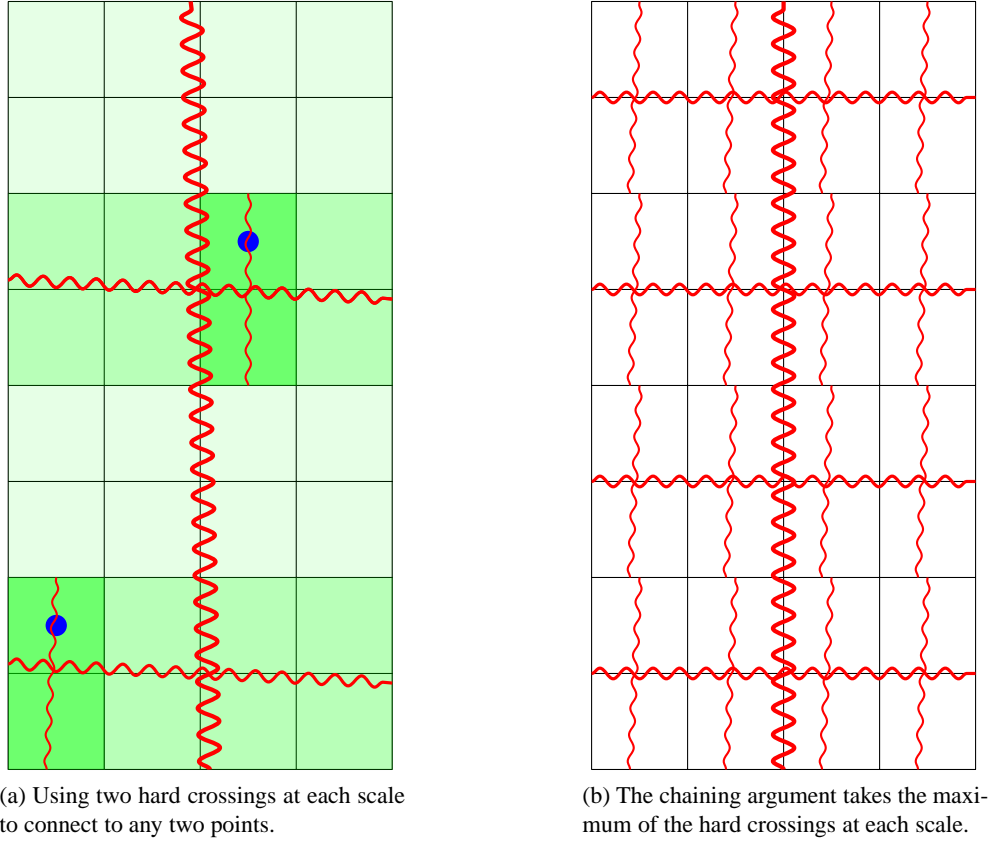


Figure 6.1

Then, since our assumption implies that the hypothesis of Theorem 5.1 holds at scale S , putting $u = 2$ we obtain

$$\begin{aligned} \mathbf{E}M_{\text{LR},S}(\mathfrak{R}) &\leq K \max \left\{ 1, 8u_0 C_{\text{PD}} \frac{\mathbf{E}\Psi_{\text{hard}}(\mathfrak{U})}{\Theta_{\text{easy}}(\mathfrak{U})[p_{\text{RSW}}]} \right\} + K^2 \left[2^{-L/3} + 2L \left(2d_{\text{pass}}^2 \sqrt{p_{\text{RSW}}} \right)^K \right] + o_{K,L}(1) \\ &\leq K \max \left\{ 1, 8u_0 C_{\text{PD}} C_{\text{RSW}} \frac{\mathbf{E}\Psi_{\text{hard}}(\mathfrak{U})}{\Theta_{\text{hard}}(\mathfrak{U})[p_{\text{RSW}}]} \right\} + K^2 \left[2^{-L/3} + 2L \left(2d_{\text{pass}}^2 \sqrt{p_{\text{RSW}}} \right)^K \right] + o_{K,L}(1) \end{aligned}$$

Finally, using the assumption that $\text{CV}^2(\Psi_{\text{hard}}(\mathfrak{U})) < \delta$, then if δ is chosen sufficiently small compared to p_{RSW} , (3.1) implies the result. \square

6.3 Diameter upper bound

We now turn our attention to the problem of estimating the point-to-point distance between two points in a box, using a chaining argument to take advantage of our good tail bound established in Proposition 6.1.

Fix a scale $S = 2^s$. Let $\mathfrak{R} = [0, S) \times [0, 2S)$. For $t \in [0, s]$ and $(i, j) \in [0, 2^t)^2$, put

$$\mathfrak{R}_{t,i,j} = \begin{cases} (i \cdot 2^{s-t}, 2 \cdot j \cdot 2^{s-t}) + [0, 2^{s-t}) \times [0, 2 \cdot 2^{s-t}) & t \text{ even} \\ (2 \cdot i \cdot 2^{s-t}, j \cdot 2^{s-t}) + [0, 2 \cdot 2^{s-t}) \times [0, 2^{s-t}) & t \text{ odd.} \end{cases}$$

For convenience, put $\mathfrak{U}_t = \mathfrak{R}_{t,0,0}$.

Proposition 6.5. *There is a $\delta = \delta_{\text{diam}} > 0$ and $C_{\text{diam}} < \infty$, independent of the scale S , so that the following holds. If*

$$\text{CV}^2(\Psi_{\text{hard}}(\mathfrak{R}_t)) < \delta \quad (6.4)$$

for all $t \geq 0$, and

$$\text{CV}^2(\Psi_{\text{easy}}(\mathfrak{R})) < \delta_{\text{RSW}} \quad (6.5)$$

for all $\mathfrak{R} \subseteq \mathfrak{R}$ of aspect ratio between $1/2$ and 2 , inclusive, then, for any $\alpha \in \mathbf{N}$ we have a $C_{\text{diamtail}}(\alpha) \geq 0$ so that, as long as γ is sufficiently small and u is sufficiently large (both compared to α),

$$\mathbf{P}(\Psi_{\text{max}}(\mathfrak{R}) \geq u\Theta_{\text{easy}}(\mathfrak{R})[q_{\text{pl}}]) \leq C_{\text{diamtail}}(\alpha)u^{-\alpha}.$$

Proof. Let $L \in \mathbf{N}$ be fixed but chosen later. By our crossing weight tail bound (6.2), applied with $L = 2^l$, $K = 2L$, and a union bound, for all $u \geq u_0$ we have

$$\mathbf{P}\left[\max_{(i,j) \in [0,2^l]^2} \Psi_{\text{hard}}(\mathfrak{R}_{t,i,j}) \geq 4uL\mathbf{E}\Psi_{\text{hard}}(\mathfrak{R}_{t+l})\right] \leq (1 + o_L(1)) \cdot 4^t \cdot u^{-L/4}, \quad (6.6)$$

Now we know that, if (6.4) holds and δ is sufficiently small (compared to p_{RSW}), then by (3.1), Theorem 5.1 (noting the hypothesis (6.5)), and Proposition 4.13 (recalling (5.1)) there is a constant C_1 (depending on δ) so that we have

$$\mathbf{E}\Psi_{\text{hard}}(\mathfrak{R}_{t+l}) \leq C_1\Theta_{\text{hard}}(\mathfrak{R}_{t+l})[p_{\text{RSW}}] \leq C_1C_{\text{RSW}}\Theta_{\text{easy}}(\mathfrak{R}_{t+l})[p_{\text{RSW}}] \leq C_1C_{\text{pl}}C_{\text{RSW}}a_{\text{pl}}^{t+l}\Theta_{\text{easy}}(\mathfrak{R})[q_{\text{pl}}]. \quad (6.7)$$

Combining (6.6) and (6.7) and putting $C = C_1C_{\text{pl}}C_{\text{RSW}}$, we get

$$\mathbf{P}\left[\max_{(i,j) \in [0,2^l]^2} \Psi_{\text{hard}}(\mathfrak{R}_{t,i,j}) \geq 4CuLa_{\text{pl}}^{\frac{t+l}{2}}\Theta_{\text{easy}}(\mathfrak{R})[q_{\text{pl}}]\right] \leq (1 + o_L(1)) \cdot 4^t \cdot u^{-L/3} \cdot a_{\text{pl}}^{L(t+l)/6},$$

so (using (2.3))

$$\begin{aligned} & \mathbf{P}\left[(\exists t) \max_{(i,j) \in [0,2^l]^2} \Psi_{\text{hard}}(\mathfrak{R}_{t,i,j}; Y_{\mathfrak{R}}) \geq 8CuLa_{\text{pl}}^{\frac{1}{4}(t+l)}\Theta_{\text{easy}}(\mathfrak{R})[q_{\text{pl}}]\right] \\ & \leq (1 + o_L(1)) \sum_{t=0}^s \left[4^t \cdot u^{-L/6} \cdot a_{\text{pl}}^{L(t+l)/6} + 4^t \exp\left(-\Omega(1) \cdot \frac{(\log(\sqrt{u}a_{\text{pl}}^{-\frac{t+l}{4}}))^2}{\log 4^t}\right) \right] \\ & \leq (1 + o_L(1)) \sum_{t=0}^s \left[4^t \cdot u^{-L/6} \cdot a_{\text{pl}}^{L(t+l)/6} + \exp\left(t \log 4 - \Omega(1) \cdot \left[\log u + \frac{(t+l)^2}{t}\right]\right) \right] \\ & = O_L(1)u^{-L/6}, \end{aligned} \quad (6.8)$$

as long as L is chosen large enough and γ small enough so that $u^{L/6}$ times the second sum is bounded in s .

Now for $x \in \mathfrak{R}$ and $t \in [0, s]$, let $\mathfrak{R}_t(x)$ be the $\mathfrak{R}_{t,i,j}$ containing x . Then

$$\Psi_{x,y}(\mathfrak{R}) \leq \sum_{t \in [0,s]} \Psi_{\text{hard}}(\mathfrak{R}_t(x); Y_{\mathfrak{R}}) + \sum_{t \in [1,s]} \Psi_{\text{hard}}(\mathfrak{R}_t(y); Y_{\mathfrak{R}}).$$

(See Figure 6.1a.) This means that

$$\Psi_{\text{max}}(\mathfrak{R}) \leq 2 \sum_{t \in [0,s]} \max_{(i,j) \in [0,2^l]^2} \Psi_{\text{hard}}(\mathfrak{R}_{t,i,j}; Y_{\mathfrak{R}}).$$

This is the chaining argument illustrated in Figure 6.1b. Applying (6.8), this implies

$$\mathbf{P} \left[\Psi_{\max}(\mathfrak{R}) \geq 8CuL\Theta_{\text{easy}}(\mathfrak{R})[q_{\text{pl}}] \sum_{t=0}^s a_{\text{pl}}^{\frac{1}{4}(t+l)} \right] \leq O_L(1) \cdot u^{-L/6}.$$

The sum is bounded so we obtain

$$\mathbf{P} \left[\Psi_{\max}(\mathfrak{R}) \geq u\Theta_{\text{easy}}(\mathfrak{R})[q_{\text{pl}}] \right] \leq O_L(1) \cdot u^{-L/6},$$

and the result follows since L can be chosen to be arbitrarily large. \square

7 Variation upper bounds

7.1 Crossing variance

Let $\mathfrak{R} = [0, KS) \times [0, LS)$.

For each edge e in the nearest-neighbor graph on \mathfrak{R} , let $\xi(e)$ and $\xi'(e)$ be independent normal random variables, all independent from those corresponding to other edges. Let

$$\xi^{\mathfrak{B}}(e) = \begin{cases} \xi(e) & e \notin \mathfrak{B} \\ \xi'(e) & e \in \mathfrak{B}. \end{cases}$$

Then, as in [LP16, (2.25)] we have the alternative definition of Gaussian free field on \mathfrak{R} as

$$Y(x) = \sum_e i_x(e) \xi(e),$$

where $i_x(e)$ is the flow through e of a unit electric current from x to $\partial\mathfrak{R}$, where the lattice is treated as an electrical network with unit resistance on each edge. Let

$$Y^{\mathfrak{B}}(x) = \sum_e i_x(e) \xi^{\mathfrak{B}}(e),$$

which is the Gaussian free field Y with the weights for edges in \mathfrak{B} resampled.

We can divide \mathfrak{R} into KL disjoint dyadic $S \times S$ sub-boxes, which we will label $\mathfrak{C}_1, \dots, \mathfrak{C}_{KL}$ in arbitrary order.

Theorem 7.1. *For any $\beta > 0$, there is a $\delta = \delta_{\text{Var}} > 0$ and a constant $C_{\text{Var}} < \infty$ so that if S, K, L are sufficiently large and γ is sufficiently small (independent of the scale S), and $\text{CV}^2(\Psi_{\text{easy}}(\mathfrak{A})) < \delta$ whenever $\mathfrak{A} \subseteq [0, 3S)^2$ has aspect ratio between $1/2$ and 2 , inclusive, then*

$$(1 - o_{K,L}(1)) \cdot \text{Var}(\Psi_{\text{LR}}(\mathfrak{R})) - o_{K,L}(1) (\mathbf{E}\Psi_{\text{LR}}(\mathfrak{R}))^2 \leq C_{\text{Var}} \cdot K \cdot L^{2/\beta} \cdot \left(\mathbf{E}\Theta_{\text{easy}}([0, 3S)^2) \right)^2. \quad (7.1)$$

Put $\mathfrak{D}_i = \overline{\mathfrak{C}_i} \cap \mathfrak{R}$. Now if $x \notin \mathfrak{D}_i$, put $\Xi(x, i) = Y(x) - Y^{\mathfrak{C}_i}(x)$. We have

$$\mathbf{E}\Xi(x, i)^2 = \mathbf{E}(Y(x) - Y^{\mathfrak{C}_i}(x))^2 = \sum_{e \in \mathfrak{C}_i} (i_x(e))^2. \quad (7.2)$$

By [LP16, Proposition 2.2], we have

$$i_x(e) = \frac{G_{\partial\mathfrak{R}}(x, e_+)}{\deg(e_+)} - \frac{G_{\partial\mathfrak{R}}(x, e_-)}{\deg(e_-)},$$

so

$$|i_x(e)| \leq |G_{\partial\mathfrak{R}}(x, e_+) - G_{\partial\mathfrak{R}}(x, e_-)|,$$

where e_- and e_+ denote the two endpoints of e and $G_{\partial\mathfrak{R}}$ denotes the Green's function for simple random walk stopped on the boundary of \mathfrak{R} . But by [LL10, Proposition 4.6.2(b), Theorem 4.4.4], we have

$$G_{\partial\mathfrak{R}}(x, y) = \mathbf{E}^x[a(Q_{\tau_{\partial\mathfrak{R}}}, y)] - a(x, y),$$

where $\{Q_t\}$ is a simple random walk, $\tau_{\partial\mathfrak{R}}$ is the hitting time of $\partial\mathfrak{R}$, \mathbf{E}^x is the expectation with respect to the law of $\{Q_t\}$ started at x , and

$$a(x, y) = \frac{2}{\pi} \log |x - y| + \frac{2C + \log 8}{\pi} + O(|x - y|^{-2}),$$

where $C \approx 0.577$ is the Euler–Mascheroni constant (usually denoted γ). This implies that (using the notation $|x - e| = \min\{|x - e_+|, |x - e_-|\}$)

$$\begin{aligned} |i_x(e)| &\leq |\mathbf{E}^x[a(Q_{\tau_{\partial\mathfrak{R}}}, e_+) - a(Q_{\tau_{\partial\mathfrak{R}}}, e_-)]| + |a(x, e_+) - a(x, e_-)| \\ &= \frac{2}{\pi} \left| \mathbf{E}^x \left[\log \left| \frac{Q_{\tau_{\partial\mathfrak{R}}} - e_+}{Q_{\tau_{\partial\mathfrak{R}}} - e_-} \right| \right] \right| + O(|Q_{\tau_{\partial\mathfrak{R}}} - e|^{-2}) + \frac{2}{\pi} \left| \log \left| \frac{x - e_+}{x - e_-} \right| \right| + O(|x - e|^{-2}) \\ &\leq \frac{2}{\pi} \mathbf{E}^x \left[\left| \frac{1}{Q_{\tau_{\partial\mathfrak{R}}} - e_-} \right| \right] + O(|Q_{\tau_{\partial\mathfrak{R}}} - e|^{-2}) + \frac{2}{\pi} \frac{1}{|x - e|} + O(|x - e|^{-2}) \\ &\leq \frac{2}{\pi S} \left(1 + \frac{1}{K \wedge L} \right) + \frac{O(1)}{S^2} \left(\frac{1}{(K \wedge L)^2} + 1 \right) \\ &\leq \frac{2}{S} \end{aligned} \tag{7.3}$$

as long as S is sufficiently large. Combining (7.2) and (7.3) yields

$$\mathbf{E}\Xi(x, i)^2 \leq 8. \tag{7.4}$$

If π is a path not passing through \mathfrak{D}_i , then

$$\psi(\pi; Y) - \psi(\pi; Y^{\mathfrak{C}_i}) = \sum_{x \in \pi} [e^{\gamma Y(x)} - e^{\gamma Y^{\mathfrak{C}_i}(x)}] = \sum_{x \in \pi} [e^{\gamma Y(x)} - e^{\gamma Y^{\mathfrak{C}_i}(x)}] = \sum_{x \in \pi} e^{\gamma Y(x)} [1 - e^{\gamma \Xi(x, i)}]. \tag{7.5}$$

We will use the following standard result.

Theorem 7.2 (Efron–Stein). *Let $X_1, \dots, X_r, X'_1, \dots, X'_r$ be independent random variables so that X_j and X'_j are identically distributed for each j , and $f : \mathfrak{R}^r \rightarrow \mathfrak{R}$. Then*

$$\text{Var}(f(X_1, \dots, X_r)) \leq \frac{1}{2} \sum_{j=1}^r \mathbf{E} \left(f(X_1, \dots, X_r) - f(X_1, \dots, X_{j-1}, X'_j, X_{j+1}, \dots, X_r) \right)^2.$$

By Theorem 7.2 (Efron–Stein), we have

$$\text{Var}(\Psi_{\text{LR}}(\mathfrak{R})) \leq \frac{1}{2} \sum_{i=1}^{KL} \mathbf{E}[\Psi_{\text{LR}}(\mathfrak{R}) - \Psi_{\text{LR}}(\mathfrak{R}; Y^{\mathfrak{C}_i})]^2. \tag{7.6}$$

We will bound the terms on the right-hand side of this inequality through the following lemma.

Lemma 7.3. *For each i , let E_i be the event that $\pi_{\text{LR}}(\mathfrak{R}) \cap \mathfrak{D}_i \neq \emptyset$. Then we have*

$$\mathbf{E} \left[\Psi_{\text{LR}}(\mathfrak{R}; Y^{\mathfrak{C}_i}) - \Psi_{\text{LR}}(\mathfrak{R}) \right]^2 \leq 4 \mathbf{E} \left(\Psi_{\partial}(\mathfrak{D}_i; Y^{\mathfrak{C}_i}) \mathbf{1}_{E_i} \right)^2 + o_{K,L}(1) \mathbf{E} \Psi_{\text{LR}}(\mathfrak{R})^2.$$

Proof. To begin, note that since Y and $Y^{\mathbb{C}_i}$ are exchangeable, we have

$$\mathbf{E}[\Psi_{\text{LR}}(\mathfrak{R}) - \Psi_{\text{LR}}(\mathfrak{R}; Y^{\mathbb{C}_i})]^2 = 2\mathbf{E}[0 \vee (\Psi_{\text{LR}}(\mathfrak{R}) - \Psi_{\text{LR}}(\mathfrak{R}; Y^{\mathbb{C}_i}))]^2. \quad (7.7)$$

Let $\pi = \pi_{\text{LR}}(\mathfrak{R})$. On the occurrence of E_i , put $\pi = \pi_0 \cup \pi_1$, where π_0 is the part of π between the first time π enters \mathfrak{D}_i and the last time π exits \mathfrak{D}_i , and π_1 is the (non-contiguous) set of all other vertices of π .

Let x^* and y^* be the first and last vertices of π_j , respectively. Note that

$$\Psi_{\text{LR}}(\mathfrak{R}; Y^{\mathbb{C}_i}) = \inf_{\pi'} \sum_{x \in \pi'} \exp(\gamma Y^{\mathbb{C}_i}(x)).$$

We claim that

$$\Psi_{\text{LR}}(\mathfrak{R}; Y^{\mathbb{C}_i}) - \Psi_{\text{LR}}(\mathfrak{R}) \leq \sum_{x \in \pi_1} e^{\gamma Y(x)} [1 - e^{\gamma \Xi(x, i)}] + \Psi_{\partial}(\mathfrak{D}_i; Y^{\mathbb{C}_i}) \mathbf{1}_{E_i}; \quad (7.8)$$

We prove (7.8) by considering two cases.

Case 1. If E does not occur, then, using (7.5) we can write

$$\inf_{\pi'} \sum_{x \in \pi'} \exp(\gamma Y^{\mathbb{C}_i}(x)) \leq \sum_{x \in \pi} \exp(\gamma Y^{\mathbb{C}_i}(x)) = \Psi_{\text{LR}}(\mathfrak{R}) + \sum_{x \in \pi} e^{\gamma Y(x)} [1 - e^{\gamma \Xi(x, i)}].$$

Case 2. Otherwise, we have

$$\Psi_{\text{LR}}(\mathfrak{R}; Y^{\mathbb{C}_i}) = \inf_{\pi'} \sum_{x \in \pi'} \exp(\gamma Y^{\mathbb{C}_i}(x)) \leq \psi(\pi_0; Y^{\mathbb{C}_i}) + \Psi_{x^*, y^*}(\mathfrak{D}_i; Y^{\mathbb{C}_i}),$$

where π' ranges over all left-right crossings of \mathfrak{R} . Therefore,

$$\begin{aligned} \Psi_{\text{LR}}(\mathfrak{R}; Y^{\mathbb{C}_i}) - \Psi_{\text{LR}}(\mathfrak{R}; Y) &\leq \psi(\pi_0; Y^{\mathbb{C}_i}) + \Psi_{x^*, y^*}(\mathfrak{D}_i; Y^{\mathbb{C}_i}) - \psi(\pi_1; Y^{\mathbb{C}_i}) - \psi(\pi_1; Y) \\ &\leq \Psi_{\partial}(\mathfrak{D}_i; Y^{\mathbb{C}_i}) + \psi(\pi_1; Y^{\mathbb{C}_i}) - \psi(\pi_1; Y) \\ &\leq \Psi_{\partial}(\mathfrak{D}_i; Y^{\mathbb{C}_i}) + \sum_{x \in \pi_1} e^{\gamma Y(x)} [1 - e^{\gamma \Xi(x, i)}]. \end{aligned}$$

By (7.4), the second term is bounded by $o_{K,L}(1) \cdot \Psi_{\text{LR}}(\mathfrak{R})$. Thus (7.8) follows from (7.7) and Cauchy-Schwarz. \square

Proof of Theorem 7.1. Let $q' \in (q_{\text{pl}}, 1)$. Note that

$$\begin{aligned} \Psi_{\partial}(\mathfrak{D}_i; Y^{\mathbb{C}_i})^2 \mathbf{1}_{E_i} &= \Psi_{\partial}(\mathfrak{D}_i; Y^{\mathbb{C}_i})^2 \mathbf{1}_{E_i} \mathbf{1}\{\Psi_{\partial}(\mathfrak{D}_i; Y^{\mathbb{C}_i}) \geq u \Theta_{\text{easy}}(\mathfrak{D}_i; Y^{\mathbb{C}_i})[q']\} \\ &\quad + \Psi_{\partial}(\mathfrak{D}_i; Y^{\mathbb{C}_i})^2 \mathbf{1}_{E_i} \mathbf{1}\{\Psi_{\partial}(\mathfrak{D}_i; Y^{\mathbb{C}_i}) < u \Theta_{\text{easy}}(\mathfrak{D}_i; Y^{\mathbb{C}_i})[q']\} \\ &\leq \Psi_{\partial}(\mathfrak{D}_i; Y^{\mathbb{C}_i})^2 \mathbf{1}\{\Psi_{\partial}(\mathfrak{D}_i; Y^{\mathbb{C}_i}) \geq u \Theta_{\text{easy}}(\mathfrak{D}_i; Y^{\mathbb{C}_i})[q']\} + u^2 \Theta_{\text{easy}}(\mathfrak{D}_i; Y^{\mathbb{C}_i})[q']^2 \mathbf{1}_{E_i}. \end{aligned} \quad (7.9)$$

Moreover, we have by (2.2) and Proposition 6.5, as long as u is sufficiently large,

$$\begin{aligned} \mathbf{E} \left[\Psi_{\partial}(\mathfrak{D}_i; Y^{\mathbb{C}_i})^2 \mathbf{1}\{\Psi_{\partial}(\mathfrak{D}_i; Y^{\mathbb{C}_i}) \geq u \Theta_{\text{easy}}(\mathfrak{D}_i; Y^{\mathbb{C}_i})[q']\} \right] &\leq (1 + o(1)) \mathbf{E} \left[\Psi_{\partial}(\mathfrak{D}_i)^2 \mathbf{1}\{\Psi_{\partial}(\mathfrak{D}_i) \geq \frac{1}{2} u \Theta_{\text{easy}}(\mathfrak{D}_i)[q_{\text{pl}}]\} \right] \\ &\leq O_{\alpha}(1) \cdot \Theta_{\text{easy}}(\mathfrak{D}_i)[q_{\text{pl}}]^2 \cdot \int_{u/2}^{\infty} v^{2-\alpha} dv = O_{\alpha}(1) \cdot \Theta_{\text{easy}}(\mathfrak{D}_i)[q_{\text{pl}}]^2 \cdot u^{3-\alpha}. \end{aligned} \quad (7.10)$$

Also, by (3.2), as long as δ is sufficiently small we have

$$\Theta_{\text{easy}}(\mathfrak{D}_i; Y^{\mathbb{C}_i})[q']^2 \leq O(1) \cdot \Theta_{\text{easy}}(\mathfrak{D}_i; Y^{\mathbb{C}_i})[q_{\text{pl}}]^2. \quad (7.11)$$

Combining Lemma 7.3, (7.9), (7.10), (7.11), and Proposition 6.4, and assuming that K and L are sufficiently large and δ, γ sufficiently small, we have

$$\begin{aligned} \frac{1}{2} \sum_{i=1}^{KL} \mathbf{E}[\Psi_{\text{LR}}(\mathfrak{R}; Y) - \Psi_{\text{LR}}(\mathfrak{R}; Y^{\mathfrak{C}_i})]^2 &\leq \frac{1}{2} \sum_{i=1}^{KL} 4 \left(\mathbf{E}(\Psi_{\partial}(\mathfrak{D}_i; Y^{\mathfrak{C}_i})^2 \mathbf{1}_{E_i}) + o_{K,L}(1) \mathbf{E}\Psi_{\text{LR}}(\mathfrak{R})^2 \right) \\ &\leq \left[\sum_{i=1}^{KL} O_{\alpha}(1) \cdot \Theta_{\text{easy}}(\mathfrak{D}_i)[q_{\text{pl}}]^2 \cdot u^{3-\alpha} \right] + \frac{1}{18} u^2 \Theta_{\text{easy}}([0, 3S]^2)[q_{\text{pl}}]^2 \mathbf{E}M_{\text{LR};S}(\mathfrak{R}) + o_{K,L}(1) \mathbf{E}\Psi_{\text{LR}}(\mathfrak{R})^2 \\ &\leq O_{\beta}(1) \cdot KL \cdot \Theta_{\text{easy}}([0, 3S]^2)[q_{\text{pl}}]^2 \cdot u^{-\beta} + \frac{C_{\text{CL}}}{9} \cdot K \cdot u^2 \cdot \Theta_{\text{easy}}([0, 3S]^2)[q_{\text{pl}}]^2 + o_{K,L}(1) \mathbf{E}\Psi_{\text{LR}}(\mathfrak{R})^2, \end{aligned}$$

where $\beta = \alpha - 3$. Then if we put $u = L^{1/\beta}$, then we obtain

$$\begin{aligned} \text{Var}(\Psi_{\text{LR}}(\mathfrak{R})) &\leq \frac{1}{2} \sum_{i=1}^{KL} \mathbf{E}[\Psi_{\text{LR}}(\mathfrak{R}) - \Psi_{\text{LR}}(\mathfrak{R}; Y^{\mathfrak{C}_i})]^2 \\ &\leq K \Theta_{\text{easy}}([0, 3S]^2)[q_{\text{pl}}]^2 \cdot \left(O_{\beta}(1) + \frac{C_{\text{CL}}}{9} \cdot L^{2/\beta} \right) + o_{K,L}(1) \mathbf{E}\Psi_{\text{LR}}(\mathfrak{R})^2. \end{aligned}$$

Then (7.1) follows from another application of (3.1). \square

7.2 Coefficient of variation

We are now ready to work towards a proof of Theorem 3.1.

Lemma 7.4. *There is a $\delta_0 > 0$ so that if $0 < \delta < \delta_0$ then the following holds. Fix a scale $S = 2^s$. Suppose that*

$$\text{CV}^2(\Psi_{\text{LR}}(\mathfrak{A})) < \delta$$

for all dyadic boxes $\mathfrak{A} \subseteq [0, S) \times [0, 2S)$. If K is sufficiently large compared to δ and $K/2 \leq L \leq 2K$ and γ is sufficiently small compared to δ , K , and L , then if $\mathfrak{R} = [0, KS) \times [0, LS)$, we have

$$\text{CV}^2(\Psi_{\text{LR}}(\mathfrak{R})) < \delta.$$

Proof. By Theorem 7.1, if K and L are sufficiently large, we have

$$(1 - o_{K,L}(1)) \cdot \text{Var}(\Psi_{\text{LR}}(\mathfrak{R})) - o_{K,L}(1) (\mathbf{E}\Psi_{\text{LR}}(\mathfrak{R}))^2 \leq C_{\text{Var}} \cdot K \cdot L^{2/\beta} \cdot \left(\mathbf{E}\Theta_{\text{easy}}([0, 3S]^2; Y^{\mathfrak{C}_i}) \right)^2.$$

Moreover, by Corollary 4.12, we have

$$\mathbf{E}\Psi_{\text{LR}}(\mathfrak{R}) \geq \frac{K}{2u_0} \Theta_{\text{easy}}(\mathfrak{A})[p_{\text{RSW}}] \cdot \left(1 - 2L \left(2d_{\text{pass}}^2 \sqrt{p_{\text{RSW}}} \right)^K - o_{K,L}(1) \right),$$

so (again recalling (5.1)) if K and L are sufficiently large and γ is sufficiently small then we have

$$\mathbf{E}\Psi_{\text{LR}}(\mathfrak{R}) \geq \frac{K}{4u_0} \Theta_{\text{easy}}(\mathfrak{A})[p_{\text{RSW}}].$$

Therefore, we have, for constants C, C' ,

$$\text{CV}^2(\Psi_{\text{LR}}(\mathfrak{R})) = \frac{\text{Var}(\Psi_{\text{LR}}(\mathfrak{R}))}{(\mathbf{E}\Psi_{\text{LR}}(\mathfrak{R}))^2} \leq \frac{C_{\text{Var}} K L^{2/\beta} \left(\mathbf{E}\Theta_{\text{easy}}(\mathfrak{A}) \right)^2}{(1 - o_{K,L}(1)) \cdot K^2 \left(\Theta_{\text{easy}}(\mathfrak{A})[p_{\text{RSW}}] \right)} + o_{K,L}(1) \leq \frac{C' L^{2/\beta}}{K} + o_{K,L}(1).$$

So if we choose K sufficiently large compared to δ and β sufficiently large then we obtain $\text{CV}^2(\Psi_{\text{LR}}(\mathfrak{R})) < \delta$ for all $K/2 \leq L \leq 2K$. \square

Lemma 7.5. For a fixed scale S_0 , we have $\text{CV}^2(\Psi_{\text{LR}}(\mathfrak{A})) = o_{S_0}(1)$ for all $\mathfrak{A} \subset [0, S_0] \times [0, 2S_0]$.

Proof. Without loss of generality, let $\mathfrak{A} = [0, S] \times [0, T]$. We have $\Psi_{\text{LR}}(\mathfrak{A}) \leq \psi(\pi_0; Y_{\mathfrak{A}})$, where π_0 is a straight-line path across \mathfrak{A} . Therefore,

$$\mathbf{E}\Psi_{\text{LR}}(\mathfrak{A})^2 \leq \mathbf{E}\psi(\pi_0; Y_{\mathfrak{A}})^2 = S^2 + o_{S_0}(1).$$

On the other hand,

$$\Psi_{\text{LR}}(\mathfrak{A}) \geq S \min_{x \in \mathfrak{A}} \exp(\gamma Y(x)),$$

so

$$\mathbf{E}\Psi_{\text{LR}}(\mathfrak{A}) \geq \mathbf{E} \left[\min_{x \in \mathfrak{A}} \exp(\gamma Y(x)) \right] = S + o_{S_0}(1).$$

Therefore,

$$\text{CV}^2(\Psi_{\text{LR}}(\mathfrak{A})) \leq \frac{\mathbf{E}\Psi_{\text{LR}}(\mathfrak{A})^2 - (\mathbf{E}\Psi_{\text{LR}}(\mathfrak{A}))^2}{(\mathbf{E}\Psi_{\text{LR}}(\mathfrak{A}))^2} = o_{S_0}(1). \quad \square$$

We have now assembled all of the pieces necessary for the proof of the main theorem.

Proof of Theorem 3.1. Apply Lemma 7.5 for some $S_0 > K$, with K chosen large enough compared to δ to satisfy the assumptions of Lemma 7.4. Then inductively applying Lemma 7.4 allows us to bound the coefficient of variation of every box of the given aspect ratios. \square

8 Subsequential limits of FPP metrics

All of the necessary estimates in hand, we now proceed to establish existence and continuity properties of the scaling limit metrics of Liouville FPP.

8.1 Tightness and subsequential convergence

As a corollary of Theorem 3.1, we will derive a tightness result for the first-passage percolation metric, properly scaled.

For each $s \in \mathbb{N}$, let $\mathfrak{R}_s = [0, s]^2$. Let $S = 2^s$. For $x, y \in [0, 1]_{\mathbb{R}}^2 \cap \frac{1}{2^s} \mathbb{Z}^2$, let

$$d_s(x, y) = \frac{\Psi_{Sx, Sy}(\mathfrak{R}_s)}{\Theta_{\text{easy}}(\mathfrak{R}_s)[q_{\text{pl}}]}.$$

For arbitrary $x, y \in [0, 1]_{\mathbb{R}}^2$, define $d_s(x, y)$ by linear interpolation.

Theorem 8.1. If γ is sufficiently small, then the sequence $\{d_s\}_{s \in \mathbb{N}}$ is tight in the Gromov–Hausdorff topology.

Note that the first part of Theorem 1.1 follows from Theorem 8.1 by Prokhorov’s theorem.

Proposition 8.2. There exists $\xi > 0$ so that, if γ is sufficiently small then for any $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that, for each $S = 2^s$, we have

$$\mathbf{P} \left(\text{there exists a dyadic square } \mathfrak{C} \subset [0, 1]_{\mathbb{R}}^2 \text{ s.t. } \text{diam}_{d_s}(\mathfrak{C} \cap \frac{1}{S} \mathbb{Z}^2) \geq C(\varepsilon) \cdot (\text{diam}_{\|\cdot\|_{\infty}} \mathfrak{C})^{\xi} \right) \leq \varepsilon,$$

where $\|\cdot\|_{\infty}$ denotes the max norm.

Proof. Let $\mathfrak{B} = [0, S]^2$ and let \mathfrak{C} be a dyadic $T \times T$ square contained in \mathfrak{B} where $T = 2^t$. By Proposition 6.5, as long as δ is sufficiently small (and γ is chosen small enough, in particular so that Theorem 3.1 holds for δ) we have a C (independent of the scale) so that

$$\mathbf{P}\left(\Psi_{\max}(\mathfrak{C}) \geq u \Theta_{\text{easy}}(\mathfrak{C})[q_{\text{pl}}]\right) \leq Cu^{-\alpha}.$$

for any dyadic square $\mathfrak{C} \subset \mathfrak{B}$. This means that, using Proposition 4.13 and (3.2), we have

$$\mathbf{P}\left(\Psi_{\max}(\mathfrak{C}) \geq u \Theta_{\text{easy}}(\mathfrak{B})[q_{\text{pl}}]\right) = \mathbf{P}\left(\Psi_{\max}(\mathfrak{C}) \geq u \frac{\Theta_{\text{easy}}(\mathfrak{B})[q_{\text{pl}}]}{\Theta_{\text{easy}}(\mathfrak{C})[q_{\text{pl}}]} \cdot \Theta_{\text{easy}}(\mathfrak{C})[q_{\text{pl}}]\right) \leq C'_\alpha u^{-\alpha} a_{\text{pl}}^{\alpha(s-t)}.$$

Putting

$$u = v a_{\text{pl}}^{\beta(s-t)}$$

for some $\beta \in (0, 1)$ to be chosen, this yields

$$\mathbf{P}\left(\Psi_{\max}(\mathfrak{C}) \geq v a_{\text{pl}}^{\beta(s-t)} \Theta_{\text{easy}}(\mathfrak{B})[q_{\text{pl}}]\right) \leq C'_\alpha v^{-\alpha} a_{\text{pl}}^{\alpha(1-\beta)(s-t)}.$$

Moreover, we have, for $0 < \beta' < \beta$, (using (2.3))

$$\begin{aligned} & \mathbf{P}\left(\Psi_{\max}(\mathfrak{C}; Y_{\mathfrak{B}}) \geq v a_{\text{pl}}^{\beta'(s-t)} \Theta_{\text{easy}}(\mathfrak{B})[q_{\text{pl}}]\right) \\ & \leq \mathbf{P}\left(\Psi_{\max}(\mathfrak{C}) \geq \sqrt{v} a_{\text{pl}}^{\beta(s-t)} \Theta_{\text{easy}}(\mathfrak{B})[q_{\text{pl}}]\right) + \exp\left(-\Omega(1) \cdot \frac{(\log(\sqrt{v} \cdot a_{\text{pl}}^{(\beta-\beta')(s-t)}))^2}{s-t}\right) \\ & \leq C'_\alpha v^{-\alpha} a_{\text{pl}}^{\alpha(1-\beta)(s-t)} + \exp\left(-\Omega(1) \cdot [(\beta-\beta')^2(s-t) + \log v]\right) \end{aligned}$$

Therefore, using a union bound, we have

$$\begin{aligned} & \mathbf{P}\left(\text{there exists a dyadic square } \mathfrak{C} \subset \mathfrak{B} \text{ s.t. } \Psi_{\max}(\mathfrak{C}; X) \geq v a_{\text{pl}}^{\beta'(s-t)} \Theta_{\text{easy}}(\mathfrak{B})[q_{\text{pl}}]\right) \\ & \leq C'_\alpha v^{-\alpha} \sum_{t=0}^s 4^{s-t} \left(a_{\text{pl}}^{\alpha(1-\beta)(s-t)} + \exp\left(-\Omega(1) \cdot [(\beta-\beta')^2(s-t) + \log v]\right) \right). \end{aligned}$$

If we choose α large enough and γ small enough (but both fixed), then the sum on the right is bounded in s , and so the right-hand side can be made arbitrarily small, uniformly in s , by increasing v . Now note that

$$a_{\text{pl}}^{\beta'(s-t)} = e^{-\beta' \log_2(T/S) \log a_{\text{pl}}} = e^{-\beta' \log(T/S) \log_2 a_{\text{pl}}} = (T/S)^{\beta' \log_2(1/a_{\text{pl}})}.$$

Therefore,

$$\mathbf{P}\left(\exists \text{ dyadic square } \mathfrak{C} \subset [0, 1]_{\mathbf{R}}^2 \text{ of side-length } \geq 1/S \text{ s.t. } \text{diam}_{d_s}(\mathfrak{C} \cap \frac{1}{S} \mathbf{Z}^2) \geq v (\text{diam}_{\|\cdot\|_\infty} \mathfrak{C})^{\beta' \log_2(1/a_{\text{pl}})}\right) \leq C''_\alpha v^{-\alpha}$$

independent of S , so we are done (with $\xi = \beta' \log_2(1/a_{\text{pl}})$ and $C(\varepsilon) = v$ chosen small enough so that $C''_\alpha v^{-\alpha} < \varepsilon$). \square

Corollary 8.3. *Let $\varepsilon > 0$. If γ is sufficiently small then there exist $C(\varepsilon), \xi(\varepsilon) > 0$ such that, for each $S = 2^s$, we have*

$$\mathbf{P}\left(\text{there exists a dyadic square } \mathfrak{C} \subset [0, 1]_{\mathbf{R}}^2 \text{ s.t. } \text{diam}_{d_s}(\mathfrak{C}) \geq C(\varepsilon) \cdot (\text{diam}_{\|\cdot\|_\infty} \mathfrak{C})^\xi\right) \leq \varepsilon.$$

Proof. Hölder conditions are preserved under linear interpolation. \square

Corollary 8.4. *Let $\varepsilon > 0$. If γ is sufficiently small then there exists $C'(\varepsilon) > 0$ such that, for each $S = 2^s$, we have*

$$\mathbf{P}\left(\text{there exist } x, y \in [0, 1]_{\mathbf{R}}^2 \text{ s.t. } d_s(x, y) \geq C'(\varepsilon) \cdot \|x - y\|_\infty^\xi\right) \leq \varepsilon$$

with ξ as above.

Proof. Any two $x, y \in [0, 1]_{\mathbf{R}}^2$ are contained within one or two adjacent dyadic boxes of side length at most twice $\|x - y\|_{\infty}$. Then the result follows from Corollary 8.3. \square

We are now ready to prove our theorem.

Proof of Theorem 8.1. By Corollary 8.4 and the compact embedding of Hölder spaces, for each $\varepsilon > 0$ and $\xi' < \xi$ there is a compact set A_ε in the Hölder- ξ' topology of Hölder- ξ functions on $[0, 1]^4$ so that $\mathbf{P}(d_s \notin A_\varepsilon) < \varepsilon$. Since the Gromov–Hausdorff topology is weaker than the uniform topology, which is in turn weaker than the Hölder- ξ topology (see for example [Mie14, Proposition 3.3.2]), A_ε is also compact in the Gromov–Hausdorff topology. This implies that $\{d_s\}$ is tight with respect to the Gromov–Hausdorff topology. \square

8.2 Hölder-continuity of limiting metrics

In this section we prove that $[0, 1]_{\mathbf{R}}^2$, equipped with the topology induced by any limit point metric, is homeomorphic to the $[0, 1]_{\mathbf{R}}^2$ with the standard topology by a Hölder-continuous homeomorphism with Hölder-continuous inverse. In fact, one of the necessary maps was obtained in the course of the proof in the previous section. The other direction follows from a similar chaining argument, but using lower bounds instead of upper bounds.

Proposition 8.5. *Any limit point of $\{d_s\}$ is almost surely Hölder- ξ' continuous with respect to the Euclidean metric for any $\xi' < \xi$ as in Proposition 8.2.*

Proof. Follows from the proof of Theorem 8.1. \square

Proposition 8.6. *If γ is sufficiently small, then there exists a $\xi' > 1$ so that for all $\varepsilon > 0$ there exist $C(\varepsilon), \xi' > 0$ such that if $S = 2^s$ is sufficiently large compared to R , then we have*

$$\mathbf{P}\left((\exists x_1, x_2) \in [0, 1]_{\mathbf{R}}^2 d_s(x, y) \leq \frac{1}{C(\varepsilon)} \cdot \|x - y\|_{\infty}^{\xi'}\right) \leq \varepsilon.$$

Moreover, we can take $\xi' \rightarrow 1$ as $\gamma \rightarrow 0$.

Proof. We will use the notation $S = 2^s$ and $T = 2^t$ throughout. Let $\mathfrak{R} = [0, S]^2$. Fix a scale $t < s$. Let $\mathfrak{U}_t = [0, T) \times [0, 2T)$. By Proposition 4.11, we have

$$\mathbf{P}\left[\min_{|\mathcal{P}_T(\pi)| \geq N} \psi(\pi; Y_{\mathfrak{R}}) \leq \frac{N}{2u} \Theta_{\text{easy}}(\mathfrak{U}_t)[p]\right] \leq (S/T)^2 \left[3(2d_{\text{pass}}^2 \sqrt{p})^N + \exp\left(-\Omega(1) \cdot \frac{(\log u)^2}{s-t}\right)\right].$$

Fixing $0 < \beta' < \beta < 1$ and summing over all scales and putting $N = (S/T)^{\beta} v$, $u = (S/T)^{\beta'} v^2$, this gives, whenever $u \geq u_0$,

$$\mathbf{P}\left[(\exists t \in [0, s)) \min_{|\mathcal{P}_T(\pi)| \geq (\frac{S}{T})^{\beta} v} \psi(\pi; Y_{\mathfrak{R}}) \leq \frac{(\frac{S}{T})^{\beta-\beta'}}{2v} \Theta_{\text{easy}}(\mathfrak{U}_t)[p]\right] \leq \sum_{t=0}^{s-1} \left(\frac{S}{T}\right)^2 \left[3(2d_{\text{pass}}^2 \sqrt{p})^{(S/T)^{\beta} v} + e^{-\Omega(1) \cdot (s-t+\log v)}\right].$$

As long as v is large enough and γ small enough, the last sum is summable over s and goes to 0, uniformly in s , as $v \rightarrow \infty$.

By Corollary 6.2, Theorem 3.1, and Theorem 5.1, as long as γ and δ are sufficiently small relative to p we have

$$\frac{\Theta_{\text{easy}}(\mathfrak{R})[p]}{\Theta_{\text{easy}}(\mathfrak{U}_t)[p]} \leq C(S/T)^{1+o(1)}.$$

Therefore, we obtain $\lim_{v \rightarrow 0} \mathbf{P}[E_s] = 0$ uniformly in s , where we define the event

$$E_s = \left\{(\exists t \in [0, s)) \min_{|\mathcal{P}_T(\pi)| \geq (S/T)^{\beta} v} \psi(\pi; Y_{\mathfrak{R}}) \leq \frac{1}{2v} \left(\frac{S}{T}\right)^{\beta-\beta'-1-o(1)} \Theta_{\text{easy}}(\mathfrak{R})[p]\right\}.$$

Using the normalized metric d_s , we have, for each

$$E_s \subset \left\{ (\exists t \in [0, s], x_1, x_2 \in [0, 1]_{\mathbf{R}}^2 \cap \frac{1}{s} \mathbf{Z}^2) \|x_1 - x_2\|_{\infty} \geq C_{\text{PD}} \cdot \nu \cdot (T/S)^{1-\beta} \text{ and } d_s(x_1, x_2) \leq \frac{1}{2\nu} \left(\frac{T}{S}\right)^{1-(\beta-\beta')+\alpha(1)} \right\}.$$

This means that there are constants C', C'' so that, with probability going to 1 as $\nu \rightarrow \infty$, for all $x_1, x_2 \in [0, 1]_{\mathbf{R}}^2 \cap \frac{1}{C'S} \mathbf{Z}^2$ we have

$$d_s(x_1, x_2) \geq \frac{C}{\nu^{2+\alpha+o(1)}} \|x_1 - x_2\|_{\infty}^{1+\alpha+o(1)}, \quad (8.1)$$

where $\alpha = 1 + \beta'/(1 - \beta)$. Since this property is preserved (up to constants) by the linear interpolation, we in fact have (8.1) for all $x_1, x_2 \in [0, 1]_{\mathbf{R}}^2$ and all scales s . By choosing β, β' appropriately, we can make α arbitrarily small as long as γ is small enough. This completes the proof of the proposition. \square

Proposition 8.7. *Any limit point d of $\{d_s\}$ almost surely has the property that*

$$d(x, y) \geq \frac{1}{C} \|x - y\|_{\infty}^{\xi'} \quad (8.2)$$

for some constant $\xi' \in (0, 1)$ and some (random) C .

Proof. Let

$$C_s = \sup_{x, y \in [0, 1]_{\mathbf{R}}^2} \frac{\|x - y\|_{\infty}^{\xi'}}{d_s(x, y)}.$$

By Proposition 8.6, $C_s < \infty$ almost surely, and moreover the sequence $\{C_s\}_s$ is tight. This means that the sequence $\{(d_s, C_s)\}_s$, where the space of metrics is given the uniform topology, is tight as well, so $\{(d_s, C_s)\}_s$ converges along subsequences. By the Skorohod representation theorem (noting that $C^{\infty}([0, 1]^4) \times \mathbf{R}$ is separable) we can put all of the (d_s, C_s) s on a common probability space and get almost-sure convergence along subsequences. But convergence along an almost-surely convergent subsequence preserves bounds of the form (8.2), and such a bound holds for d_s along any almost-surely convergent subsequence of $\{(d_s, C_s)\}_s$ since in such a case the C_s s will be bounded. Thus the proposition is proved. \square

The second statement of Theorem 1.1 is the combination of the results of Proposition 8.5 and Proposition 8.6.

References

- [ADH15] Antonio Auffinger, Michael Damron, and Jack Hanson, *50 years of first passage percolation*, Preprint, available at <http://arxiv.org/abs/1511.03262>.
- [Adl90] Robert J. Adler, *An introduction to continuity, extrema, and related topics for general Gaussian processes*, Institute of Mathematical Statistics Lecture Notes—Monograph Series, 12, Institute of Mathematical Statistics, Hayward, CA, 1990.
- [BDC12] Vincent Beffara and Hugo Duminil-Copin, *The self-dual point of the two-dimensional random-cluster model is critical for $q \geq 1$* , Probab. Theory Related Fields **153** (2012), no. 3-4, 511–542.
- [BDFG04] J. Bouttier, P. Di Francesco, and E. Guitter, *Planar maps as labeled mobiles*, Electron. J. Combin. **11** (2004), no. 1, Research Paper 69, 27.
- [BDZ16] Maury Bramson, Jian Ding, and Ofer Zeitouni, *Convergence in law of the maximum of the two-dimensional discrete Gaussian free field*, Comm. Pure Appl. Math. **69** (2016), no. 1, 62–123.

- [CV81] Robert Cori and Bernard Vauquelin, *Planar maps are well labeled trees*, Canad. J. Math. **33** (1981), no. 5, 1023–1042.
- [DCHN11] Hugo Duminil-Copin, Clément Hongler, and Pierre Nolin, *Connection probabilities and RSW-type bounds for the two-dimensional FK Ising model*, Comm. Pure Appl. Math. **64** (2011), no. 9, 1165–1198.
- [DCMT] Hugo Duminil-Copin, Ioan Manolescu, and Vincent Tassion, *An RSW theorem for Gaussian free field*, In preparation.
- [DCRT16] Hugo Duminil-Copin, Aran Raoufi, and Vincent Tassion, *A new computation of the critical point for the planar random-cluster model with $q \geq 1$* , Preprint, arXiv 1604.03702.
- [DCST15] Hugo Duminil-Copin, Vladas Sidoravicius, and Vincent Tassion, *Continuity of the phase transition for planar random-cluster and potts models with $1 \leq q \leq 4$* , Preprint, arXiv 1505.04159.
- [DG] Jian Ding and Subhajit Goswami, *Liouville first passage percolation: the weight exponent is strictly less than 1 at high temperature*, In preparation.
- [DG15] ———, *First passage percolation on the exponential of two-dimensional branching random walk*, arXiv:1511.06932v2 [math.PR] (2015).
- [DMS14] Bertrand Duplantier, Jason Miller, and Scott Sheffield, *Liouville quantum gravity as a mating of trees*, Preprint, available at <http://arxiv.org/abs/1409.7055>.
- [DS11] Bertrand Duplantier and Scott Sheffield, *Liouville quantum gravity and KPZ*, Invent. Math. **185** (2011), no. 2, 333–393.
- [DZ15] Jian Ding and Fuxi Zhang, *Non-universality for first passage percolation on the exponential of log-correlated gaussian fields*, Preprint, available at <http://arxiv.org/abs/1506.03293>.
- [Fer75] X. Fernique, *Régularité des trajectoires des fonctions aléatoires gaussiennes*, École d’Été de Probabilités de Saint-Flour, IV-1974, Springer, Berlin, 1975, pp. 1–96. Lecture Notes in Math., Vol. 480.
- [GK12] Geoffrey R. Grimmett and Harry Kesten, *Percolation since Saint-Flour*, Percolation theory at Saint-Flour, Probab. St.-Flour, Springer, Heidelberg, 2012, pp. ix–xxvii.
- [Led01] Michel Ledoux, *The concentration of measure phenomenon*, Mathematical Surveys and Monographs, vol. 89, American Mathematical Society, Providence, RI, 2001.
- [LG07] Jean-François Le Gall, *The topological structure of scaling limits of large planar maps*, Invent. Math. **169** (2007), no. 3, 621–670.
- [LG10] ———, *Geodesics in large planar maps and in the Brownian map*, Acta Math. **205** (2010), no. 2, 287–360.
- [LG13] ———, *Uniqueness and universality of the Brownian map*, Ann. Probab. **41** (2013), no. 4, 2880–2960.
- [LGP08] Jean-François Le Gall and Frédéric Paulin, *Scaling limits of bipartite planar maps are homeomorphic to the 2-sphere*, Geom. Funct. Anal. **18** (2008), no. 3, 893–918.
- [LL10] Gregory F. Lawler and Vlada Limic, *Random walk: A modern introduction*, Cambridge Studies in Advanced Mathematics, vol. 123, Cambridge University Press, Cambridge, 2010.

- [LP16] Russell Lyons and Yuval Peres, *Probability on trees and networks*, Cambridge University Press, 2016, Available at <http://pages.iu.edu/~rdlyons/>.
- [Mie13] Grégory Miermont, *The Brownian map is the scaling limit of uniform random plane quadrangulations*, Acta Math. **210** (2013), no. 2, 319–401.
- [Mie14] Grégory Miermont, *Aspects of random maps*, Lecture Notes of the 2014 Saint-Flour Probability Summer School, 2014, preliminary draft, available at <http://perso.ens-lyon.fr/gregory.miermont/coursSaint-Flour.pdf>.
- [MS15a] Jason Miller and Scott Sheffield, *Liouville quantum gravity and the Brownian map I: The QLE(8/3,0) metric*, Preprint, available at <http://arxiv.org/abs/1507.00719>.
- [MS15b] ———, *Quantum Loewner evolution*, Preprint, available at <http://arxiv.org/abs/1409.7055>.
- [Pit82] Loren D. Pitt, *Positively correlated normal random variables are associated*, Annals of Probability **10** (1982), no. 2, 496–499.
- [Pol81] A. M. Polyakov, *Quantum geometry of bosonic strings*, Phys. Lett. B **103** (1981), no. 3, 207–210.
- [Rus78] Lucio Russo, *A note on percolation*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **43** (1978), no. 1, 39–48.
- [Rus81] ———, *On the critical percolation probabilities*, Z. Wahrsch. Verw. Gebiete **56** (1981), no. 2, 229–237.
- [RV14] Rémi Rhodes and Vincent Vargas, *Gaussian multiplicative chaos and applications: a review*, Probab. Surv. **11** (2014), 315–392.
- [Sch88] G Schaeffer, *Conjugaison d’arbres et cartes combinatoires aléatoires*, Ph.D. thesis, Université Bordeaux I.
- [SW78] P. D. Seymour and D. J. A. Welsh, *Percolation probabilities on the square lattice*, Ann. Discrete Math. **3** (1978), 227–245, Advances in graph theory (Cambridge Combinatorial Conf., Trinity College, Cambridge, 1977).
- [Tas14] Vincent Tassion, *Crossing probabilities for Voronoi percolation*, Annals of Probability (2014), to appear, available at <http://arxiv.org/abs/1410.6773>.